

SPIN^c QUANTIZATION, PREQUANTIZATION AND CUTTING

by

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Abstract

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In this thesis we extend Lerman's cutting construction (see [4]) to spin^c-structures. Every spin^c-structure on an even-dimensional Riemannian manifold gives rise to a Dirac operator D^+ acting on sections of the associated spinor bundle. The spin^c-quantization of a spin^c-manifold is defined to be $\ker(D^+) - \operatorname{coker}(D^+)$. It is a virtual vector space, and in the presence of a Lie group action, it is a virtual representation. In [5] *signature quantization* is defined and shown to be *additive under cutting*. We prove that the spin^c-quantization of an S^1 -manifold is also additive under cutting. Our proof uses the method of localization, i.e., we express the spin^c-quantization of a manifold in terms of local data near connected components of the fixed point set.

For a symplectic manifold (M, ω) , a spin^c-prequantization is a spin^c-structure together with a connection compatible with ω . We explain how one can cut a spin^c-prequantization and show that the choice of a spin^c-structure on \mathbb{C} (which is part of the cutting process) must be compatible with the moment level set along which the cutting is performed.

Finally, we prove that the spin^c and metaplectic^c groups satisfy a universal property: Every structure that makes the construction of a spinor bundle possible must factor uniquely through a spin^c-structure in the Riemannian case, or through a metaplectic^c structure in the symplectic case.

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Chapter 1

Introduction

In this thesis we discuss spin^c -structures on manifolds and their relation to the problem of quantization. We generalize the process of *cutting* originally introduced by E. Lerman in [4].

The group $\text{Spin}(n)$ is the unique (connected) double cover of the special orthogonal group $\text{SO}(n)$. In fact, $\text{Spin}(n)$ is the universal cover of $\text{SO}(n)$ if $n \geq 3$. If $-1 \in \text{Spin}(n)$ denotes the non-trivial element in the kernel of $\text{Spin}(n) \rightarrow \text{SO}(n)$, then the complexified spin group is defined as

$$\text{Spin}^c(n) = [\text{Spin}(n) \times U(1)] / K$$

where $K = \{(1, 1), (-1, -1)\}$. Alternatively, the groups $\text{Spin}(n)$ and $\text{Spin}^c(n)$ can be defined as subgroups of the invertible elements in certain Clifford algebras.

A *spin^c-structure* on an oriented Riemannian manifold M is a lift of the structure group from $\text{SO}(n)$ to the group $\text{Spin}^c(n)$. Manifolds with such structure are called *spin^c-manifolds*. Roughly speaking, this means that the principal $\text{SO}(n)$ -bundle of oriented orthonormal frames in M is replaced by a principal $\text{Spin}^c(n)$ -bundle in a compatible way. In the presence of a Lie group action G on M , we require that the action lifts to the $\text{Spin}^c(n)$ -bundle, and thus get a *G-equivariant spin^c-structure*.

One of the reasons for being interested in spin^c -structures is that they enable us to

define *spinors*, which are elements of a certain vector bundle S over M , and are common in mathematical physics.

If the spin^c -manifold M is even dimensional, we can define a differential operator D^+ , called a *Dirac operator*, acting on sections of the spinor bundle S (this involves a choice of a connection on the spin^c -structure). The index of D^+ is called the *spin^c -quantization* of our manifold, and is a virtual vector space (or, in the equivariant case, a virtual representation of G). We assume that M is compact to assure that the kernel and co-kernel of D^+ are finite dimensional.

E. Lerman developed the process of cutting in [4]. If (M, ω) is a symplectic manifold endowed with a Hamiltonian circle action, then the cutting produces two new symplectic manifolds $(M_{\text{cut}}^{\pm}, \omega_{\text{cut}}^{\pm})$. This construction was extended to spin^c -manifolds in [6], but some details were missing. In Part I we fill in the gaps in the cutting process of a spin^c -structure and prove that spin^c -quantization is *additive under cutting*, i.e., the quantization of the original manifold M is isomorphic (as a virtual representation of S^1) to the direct sum of the quantizations of the cut spaces M_{cut}^{\pm} .

The proof of additivity uses the technique of localization: we express the quantization of a spin^c -manifold in terms of local data around connected components of the fixed point set. Those formulas are in fact modifications of Kostant formulas for almost complex quantization, so we call them *the generalized Kostant formulas*.

Note that for the more common Kähler quantization, this additivity property does not hold (see [11, page 258]). On the other hand, Guillemin, Sternberg and Weitzman proved in [5] that *signature quantization* does satisfy the additivity under cutting property. In fact, this motivated our work in Part I.

In Part I we do not assume that our manifold is symplectic. In Part II we relate our constructions from Part I to symplectic geometry. For a symplectic manifold (M, ω) we define *spin^c -prequantization* to be a spin^c -structure *and a connection* that are compatible with the two-form ω in a certain sense. We believe that this definition already incor-

porates the ‘twist by half-forms’ which is part of the Geometric Quantization scheme developed by Kostant and Souriau (See [13]).

We then extend the cutting construction developed in Part I to manifolds endowed with a spin^c -prequantization. Our main statement (Theorem 11.4.1) relates the choice of a ‘cutting surface’ to the choice of a spin^c -prequantization on the complex plane, which is part of the cutting process. Moreover, we conclude that the ‘cutting value’ must lie halfway between two consecutive integers in $\text{Lie}(S^1)^* \cong \mathbb{R}$, which explains why the additivity property holds.

We discuss in detail spin^c -structures, prequantization, and quantization for the two-sphere in Chapters 8 and 12 to illustrate our results.

Recall that in the context of Kähler and almost-complex quantization, a symplectic manifold (M, ω) is prequantizable if and only if $\frac{1}{2\pi}\omega$ represents an integral cohomology class in $H^2(M; \mathbb{R})$. For spin^c -prequantization this is no longer true in general (although it remains true when M is the two-sphere). In Chapter 13 we show that for an even n , if ω is a spin^c -prequantizable form on $\mathbb{C}P^n$, then $\frac{1}{2\pi}\omega$ will never be integral.

In Part III of the thesis we prove a universal property of the spin^c and metaplectic^c groups. Recall that the metaplectic^c is defined as

$$Mp^c(n) = [Mp(n) \times U(1)] / K$$

where $Mp(n)$ is the unique (connected) double cover of the symplectic group

$$Sp(n) = \{A \in GL(2n, \mathbb{R}) : \omega(Av, Aw) = \omega(v, w)\}$$

and $K = \{(1, 1), (-1, -1)\}$ as before.

It is known that one has to introduce additional structure on an oriented Riemannian manifold in order to construct a bundle of spinors (see [1, Introduction]). Examples of such structures are spin , spin^c and almost complex structures. Our main theorem in Chapter 17 asserts that any structure that enables the construction of spinors must factor through a spin^c -structure, in a unique way. Thus, spin^c -structures are a universal

solution to this problem. Similarly, we show that metaplectic^c-structures are a universal solution to the analogous problem for symplectic manifolds.

Each part was submitted separately for publication, and therefore material from one part is sometimes repeated in another part. I apologize for having those repetitions.

Throughout this thesis, all spaces are assumed to be smooth manifolds, and all maps and actions are assumed to be smooth. The principal action in a principal bundle will be always a right action. A real vector bundle E , equipped with a fiberwise inner product, will be called a *Riemannian vector bundle*. If the fibers are also oriented, then its bundle of oriented orthonormal frames will be denoted by $SOF(E)$. For an oriented Riemannian manifold M , we will simply write $SOF(M)$, instead of $SOF(TM)$.

Part I

Spin^c Quantization and Additivity under Cutting

Chapter 2

Introduction to Part I

In this part we discuss S^1 -equivariant spin^c structures on compact oriented Riemannian S^1 -manifolds, and the Dirac operator associated to those structures. The index of the Dirac operator is a virtual representation of S^1 , and is called *the spin^c quantization* of the spin^c manifold.

Also, we describe a cutting construction for spin^c structures. Cutting was first developed by E. Lerman for symplectic manifolds (see [4]), and then extended to manifolds that possess other structures. In particular, our recipe is closely related to the one described in [6].

The goal of this part is to point out a relation between spin^c quantization and cutting. We claim that the quantization of our original manifold is isomorphic (as a virtual representation) to the direct sum of the quantizations of the cut spaces. We refer to this property as ‘additivity under cutting’.

In [5], Guillemin, Sternberg and Weitsman define *signature quantization* and show that it satisfies ‘additivity under cutting’. In fact, this observation motivated our work.

It is important to mention that this property does not hold for the most common ‘almost-complex quantization’. In this case, we start with an almost complex compact manifold, and a Hermitian line bundle with Hermitian connection, and construct the

Dolbeaut-Dirac operator associated to this data. Its index is a virtual vector space, and in the presence of an S^1 -action on the manifold and the line bundle, we get a virtual representation of S^1 , called the Dolbeaut-Dirac quantization of the manifold (see [2] or [11]). This is a special case of our spin^c quantization, since an almost complex structure and a complex line bundle determine a spin^c -structure, which gives rise to the same Dirac operator (See Lemma 2.7 and Remark 2.9 in [6], and Appendix D in [3]). However, in the almost complex case, the cutting is done along the zero level set of the moment map determined by the line bundle and the connection. This results in additivity for all weights except zero. More precisely, if $N_{\pm}(\mu)$ denotes the multiplicity of the weight μ in the almost complex quantization of the cut spaces, and $N(\mu)$ is the weight of μ in the quantization of the original manifold, we have (see p.258 in [11])

$$N(\mu) = N_+(\mu) + N_-(\mu) \quad , \quad \mu \neq 0$$

but

$$N(0) = N_+(0) = N_-(0)$$

and therefore there is no additivity in general.

On the other hand, if spin^c cutting is done for a spin^c manifold M (in particular, the spin^c -structure can come from an almost complex structure), then the additivity will hold for any weight. Roughly speaking, this happens because the spin^c cutting is done at the level set $1/2$ of the ‘moment map’, which is not a weight (i.e., an integer) for the group S^1 .

In order to make this part as self-contained as possible, we review the necessary background on spin^c equivariant structures, Clifford algebras and spin^c quantization in Chapter 3. We describe in details the cutting process in Chapter 4. In Chapters 5 and 6 we develop Kostant-type formulas for spin^c quantizations in terms of local data around connected components of the fixed point set, and finally in Chapter 7 we prove the additivity result. In Chapter 8, we give a detailed example that illustrates the additivity

property of spin^c quantization. In particular, we classify and cut all the S^1 -equivariant spin^c structures on the two-sphere.

Chapter 3

Spin^c Quantization

In this chapter we define the concept of *spin^c quantization* as the index of the Dirac spin^c operator associated to a manifold endowed with a spin^c-structure. The quantization will be a virtual complex vector space, and in the presence of a Lie group action it will be a virtual representation of that group.

3.1 Spin^c structures

Definition 3.1.1. Let V be a finite dimensional vector space over $\mathbb{K} = \mathbb{R}$ or \mathbb{C} , equipped with a symmetric bilinear form $B : V \times V \rightarrow \mathbb{K}$. The *Clifford algebra* $Cl(V, B)$ is the quotient $T(V)/I(V, B)$ where $T(V)$ is the tensor algebra of V and $I(V, B)$ is the ideal generated by $\{v \otimes v - B(v, v) \cdot 1 : v \in V\}$.

Remark 3.1.1. If v_1, \dots, v_n is an orthogonal basis for V , then $Cl(V, B)$ is the algebra generated by v_1, \dots, v_n subject to the relations $v_i^2 = B(v_i, v_i) \cdot 1$ and $v_i v_j = -v_j v_i$ for $i \neq j$.

Note that V is a vector subspace of $Cl(V, B)$.

Definition 3.1.2. If $V = \mathbb{R}^k$ and B is minus the standard inner product on V , we define:

1. $C_k := Cl(V, B)$, and $C_k^c := Cl(V, B) \otimes \mathbb{C}$.

These are finite dimensional algebras over \mathbb{R} and \mathbb{C} , respectively.

2. The *spin group*

$$\text{Spin}(k) = \{v_1 v_2 \dots v_l : v_i \in \mathbb{R}^k, \|v_i\| = 1 \text{ and } 0 \leq l \text{ is even}\} \subset C_k$$

3. The *spin^c group*

$$\text{Spin}^c(k) = (\text{Spin}(k) \times U(1)) / K$$

where $U(1) \subset \mathbb{C}$ is the unit circle and $K = \{(1, 1), (-1, -1)\}$.

Remark 3.1.2.

1. Equivalently, one can define

$$\begin{aligned} \text{Spin}^c(k) &= \\ &= \{c \cdot v_1 \cdots v_l : v_i \in \mathbb{R}^k, \|v_i\| = 1, 0 \leq l \text{ is even, and } c \in U(1)\} \subset C_k^c \end{aligned}$$

2. The group $\text{Spin}(k)$ is connected for $k \geq 2$.

Proposition 3.1.1.

1. There is a linear map $C_k \rightarrow C_k$, $x \mapsto x^t$, characterized by $(v_1 \dots v_l)^t = v_l \dots v_1$ for all $v_1, \dots, v_l \in \mathbb{R}^k$.
2. For each $x \in \text{Spin}(k)$ and $y \in \mathbb{R}^k$, we have $xyx^t \in \mathbb{R}^k$.
3. For each $x \in \text{Spin}(k)$, the map $\lambda(x) : \mathbb{R}^k \rightarrow \mathbb{R}^k$, $y \mapsto xyx^t$, is in $SO(k)$, and $\lambda : \text{Spin}(k) \rightarrow SO(k)$ is a double covering for $k \geq 1$. It is a universal covering map for $k \geq 3$.

For the proof, see page 16 in [1].

Definition 3.1.3. Let M be a manifold and Q a principal $SO(k)$ -bundle on M . A *spin^c structure* on Q is a principal $\text{Spin}^c(k)$ -bundle $P \rightarrow M$, together with a map $\Lambda : P \rightarrow Q$, such that the following diagram commutes.

$$\begin{array}{ccc} P \times \text{Spin}^c(k) & \longrightarrow & P \\ \downarrow \Lambda \times \lambda^c & & \downarrow \Lambda \\ Q \times SO(k) & \longrightarrow & Q \end{array}$$

Here, the maps corresponding to the horizontal arrows are the principal actions, and $\lambda^c : \text{Spin}^c(k) \rightarrow SO(k)$ is given by $[x, z] \mapsto \lambda(x)$, where $\lambda : \text{Spin}(k) \rightarrow SO(k)$ is the double covering.

Remark 3.1.3.

1. A spin^c-structure on an oriented Riemannian vector bundle E is a spin^c-structure on the associated bundle of oriented orthonormal frames, $\text{SOF}(E)$.
2. A spin^c-structure on an oriented Riemannian manifold is a spin^c-structure on its tangent bundle.
3. Given a spin^c-structure on $Q \rightarrow M$, its *determinant line bundle* is $\mathbb{L} = P \times_{\text{Spin}^c(k)} \mathbb{C}$, where the left action of $\text{Spin}^c(k)$ on \mathbb{C} is given by $[x, z] \cdot w = z^2 w$. This is a Hermitian line bundle over M .

3.2 Equivariant spin^c structures

Definition 3.2.1. Let G, H be Lie groups. A *G-equivariant principal H-bundle* is a principal H -bundle $\pi : Q \rightarrow M$ together with left G -actions on Q and M , such that

1. $\pi(g \cdot q) = g \cdot \pi(q)$ for all $g \in G$, $q \in Q$
(i.e., G acts on the fiber bundle $\pi : Q \rightarrow M$).

2. $(g \cdot q) \cdot h = g \cdot (q \cdot h)$ for all $g \in G$, $q \in Q$, $h \in H$
(i.e., the actions of G and H commute).

Remark 3.2.1. It is convenient to think of a G -equivariant principal H -bundle in terms of the following commuting diagram (the horizontal arrows correspond to the G and H actions).

$$\begin{array}{ccccc}
 G \times Q & \longrightarrow & Q & \longleftarrow & Q \times H \\
 \text{Id} \times \pi \downarrow & & \downarrow \pi & & \\
 G \times M & \longrightarrow & M & &
 \end{array}$$

Definition 3.2.2. Let $\pi : E \rightarrow M$ be a fiberwise oriented Riemannian vector bundle, and let G be a Lie group. If a G -action on $E \rightarrow M$ is given that preserves the orientations and the inner products of the fibers, we will call E a *G -equivariant oriented Riemannian vector bundle*.

Remark 3.2.2.

1. If E is a G -equivariant oriented Riemannian vector bundle, then $\text{SOF}(E)$ is a G -equivariant principal $\text{SO}(k)$ -bundle, where $k = \text{rank}(E)$.
2. If a Lie group G acts on an oriented Riemannian manifold M by orientation preserving isometries, then the frame bundle $\text{SOF}(M)$ becomes a G -equivariant principal $\text{SO}(m)$ -bundle, where $m = \dim(M)$.

Definition 3.2.3. Let $\pi : Q \rightarrow M$ be a G -equivariant principal $\text{SO}(k)$ -bundle. A *G -equivariant spin^c -structure* on Q is a spin^c structure $\Lambda : P \rightarrow Q$ on Q , together with a left action of G on P , such that

1. $\Lambda(g \cdot p) = g \cdot \Lambda(p)$ for all $p \in P$, $g \in G$ (i.e., G acts on the bundle $P \rightarrow Q$).

2. $g \cdot (p \cdot x) = (g \cdot p) \cdot x$ for all $g \in G$, $p \in P$, $x \in \text{Spin}^c(k)$
 (i.e., the actions of G and $\text{Spin}^c(k)$ on P commute).

Remark 3.2.3.

1. We have the following commuting diagram (where the horizontal arrows correspond to the principal and the G -actions).

$$\begin{array}{ccccc}
 G \times P & \longrightarrow & P & \longleftarrow & P \times \text{Spin}^c(k) \\
 \text{Id} \times \Lambda \downarrow & & \Lambda \downarrow & & \Lambda \times \lambda^c \downarrow \\
 G \times Q & \longrightarrow & Q & \longleftarrow & Q \times \text{SO}(k) \\
 \text{Id} \times \pi \downarrow & & \pi \downarrow & & \\
 G \times M & \longrightarrow & M & &
 \end{array}$$

2. The bundle $P \rightarrow M$ is a G -equivariant principal $\text{Spin}^c(k)$ -bundle.
 3. The determinant line bundle $\mathbb{L} = P \times_{\text{Spin}^c(k)} \mathbb{C}$ is a G -equivariant Hermitian line bundle.

3.3 Clifford multiplication and spinor bundles

Proposition 3.3.1. *The number of inequivalent irreducible (complex) representations of the algebra $C_k^c = C_k \otimes \mathbb{C}$ is 1 if k is even and 2 if k is odd.*

For a proof, see Theorem I.5.7 in [3].

Note that, for all k , $\mathbb{R}^k \subset C_k \subset C_k^c$.

Definition 3.3.1. Let k be a positive integer. Define a *Clifford multiplication map*

$$\mu : \mathbb{R}^k \otimes \Delta_k \rightarrow \Delta_k \quad \text{by} \quad \mu(x \otimes v) = \rho_k(x)v$$

where $\rho_k : C_k^c \rightarrow \text{End}(\Delta_k)$ is an irreducible representation of C_k^c (a choice is to be made if k is odd).

Definition 3.3.2. Let k be a positive integer and ρ_k an irreducible representation of C_k^c . The restriction of ρ_k to the group $\text{Spin}(k) \subset C_k \subset C_k^c$ is called *the complex spin representation* of $\text{Spin}(k)$. It will be also denoted by ρ_k .

Remark 3.3.1. For an odd integer k , the complex spin representation is independent of the choice of an irreducible representation of C_k^c (see Proposition I.5.15 in [3]).

The following proposition summarizes a few facts about the complex spin representation. Proofs can be found in [1] and in [3].

Proposition 3.3.2. *Let $\rho_k : \text{Spin}(k) \rightarrow \text{End}(\Delta_k)$ be the complex spin representation. Then*

1. $\dim_{\mathbb{C}} \Delta_k = 2^l$, where $l = k/2$ if k is even, and $l = (k - 1)/2$ if k is odd.
2. ρ_k is a faithful representation of $\text{Spin}(k)$.
3. If k is odd, then ρ_k is irreducible.
4. If k is even, then ρ_k is reducible, and splits as a sum of two inequivalent irreducible representations of the same dimension,

$$\rho_k^+ : \text{Spin}(k) \rightarrow \text{End}(\Delta_k^+) \quad \text{and} \quad \rho_k^- : \text{Spin}(k) \rightarrow \text{End}(\Delta_k^-) .$$

Remark 3.3.2. The representation ρ_k extends to a representation of the group $\text{Spin}^c(k)$, and will be also denoted by ρ_k . Explicitly,

$$\rho_k : \text{Spin}^c(k) \rightarrow \text{End}(\Delta_k) \quad , \quad \rho_k([x, z])v = z \cdot \rho_k(x)v .$$

Definition 3.3.3. Let P be a spin^c -structure on an oriented Riemannian manifold M . Then the *spinor bundle* of the spin^c -structure is the complex vector bundle $S =$

$P \times_{\text{Spin}^c(m)} \Delta_m$, where $m = \dim(M)$.

If P is a G -equivariant spin^c -structure, then S will be a G -equivariant complex vector bundle.

Remark 3.3.3. It is possible to choose a Hermitian inner product on Δ_k which is preserved by the action of the group $\text{Spin}^c(k)$. This induces a Hermitian inner product on the spinor bundle. In the G -equivariant case, G will act on the fibers of S by Hermitian transformations.

From Proposition 3.3.2 we get

Proposition 3.3.3. *Let P be a (G -equivariant) spin^c -structure on an oriented Riemannian manifold M of even dimension, and let S be the corresponding spinor bundle. Then S splits as a sum $S = S^+ \oplus S^-$ of two (G -equivariant) complex vector bundles.*

Remark 3.3.4. If M is an oriented Riemannian manifold, equipped with a spin^c -structure, and a corresponding spinor bundle S , then a Clifford multiplication map $\mu : \mathbb{R}^k \otimes \Delta_k \rightarrow \Delta_k$ induces a map on the associated bundles $TM \otimes S \rightarrow S$. This map is also called Clifford multiplication and will be denoted by μ as well.

3.4 The spin^c Dirac operator

The following is a reformulation of Proposition D.11 from [3]:

Proposition 3.4.1. *Let M be an oriented Riemannian manifold of dimension $m \geq 1$, $P \rightarrow \text{SO}(M)$ a spin^c -structure on M , and $P_1 = P/\text{Spin}(m)$ (this quotient can be defined since $\text{Spin}(m)$ embeds naturally in $\text{Spin}^c(m)$). Then*

1. P_1 is a principal $U(1)$ -bundle over M , and $P \rightarrow \text{SO}(M) \times P_1$ is a double cover.
2. The determinant line bundle of the spin^c structure is naturally isomorphic to $\mathbb{L} = P_1 \times_{U(1)} \mathbb{C}$.

3. If $A : TP_1 \rightarrow i\mathbb{R}$ is an invariant connection, and $Z : T(\text{SOF}(M)) \rightarrow \mathfrak{so}(m)$ the Levi-Civita connection on M , then the $\text{SO}(m) \times U(1)$ -invariant connection $Z \times A$ on $\text{SOF}(M) \times P_1$ lifts to a unique $\text{Spin}^c(m)$ -invariant connection on its double cover P .

Remark 3.4.1. If G acts on M by orientation preserving isometries, P is a G -equivariant spin^c structure on M , and the connection A on P_1 is chosen to be G -invariant, then $Z \times A$ and its lift to P will be G -invariant.

Definition 3.4.1. Assume the following data are given:

1. An oriented Riemannian manifold M of dimension m .
2. A spin^c -structure $P \rightarrow \text{SOF}(M)$ on M , with the associated spinor bundle S .
3. A connection on $P_1 = P/\text{Spin}(m)$ which gives rise to a covariant derivative $\nabla : \Gamma(S) \rightarrow \Gamma(T^*M \otimes S)$.

The *Dirac spin^c operator* (or simply, the *Dirac operator*) associated to this data is the composition

$$D : \Gamma(S) \xrightarrow{\nabla} \Gamma(T^*M \otimes S) \xrightarrow{\simeq} \Gamma(TM \otimes S) \xrightarrow{\mu} \Gamma(S) ,$$

where the isomorphism is induced by the Riemannian metric (which identifies $T^*M \simeq TM$), and μ is the Clifford multiplication.

Remark 3.4.2.

1. Since there are two ways to define μ when k is odd, one has to make a choice for μ to get a well-defined Dirac operator.
2. If G acts on M by orientation preserving isometries, the spin^c -structure on M is G -equivariant, and the connection on P_1 is G -invariant, then the Dirac operator D will commute with the G -action on $\Gamma(S)$.

3. If $\dim(M)$ is even, then the Dirac operator decomposes into a sum of two operators $D^\pm : \Gamma(S^\pm) \rightarrow \Gamma(S^\mp)$ (since μ interchanges S^+ and S^-), which are also called Dirac operators.
4. If the manifold M is complete, then the Dirac operator is essentially self-adjoint on $L^2(S)$, the square integrable sections of S (See Theorem II.5.7 in [3] or Chapter 4 in [1]).

3.5 Spin^c quantization

We now restrict to the case of an even dimensional oriented Riemannian manifold M which is also compact. Since the concept of spin^c quantization will be defined as the index of the operator D^+ , it makes sense to define it only for even dimensional manifolds. The compactness is used to ensure that $\dim(\ker(D^+))$ and $\dim(\text{coker}(D^+))$ are finite.

Definition 3.5.1. Assume that the following data are given:

1. An oriented compact Riemannian manifold M of dimension $2m$.
2. G a Lie group that acts on M by orientation preserving isometries.
3. $P \rightarrow \text{SOF}(M)$ a G -equivariant spin^c -structure.
4. A $U(1)$ and G -invariant connection on $P_1 = P/\text{Spin}(2m)$.

Then the spin^c quantization of M , with respect to the above data, is the virtual complex G -representation $Q(M) = \ker(D^+) - \text{coker}(D^+)$.

The *index* of D^+ is the integer $\text{index}(D^+) = \dim(\ker(D^+)) - \dim(\text{coker}(D^+))$.

Remark 3.5.1. In the absence of a G action, the spin^c quantization is just a virtual complex vector space.

Chapter 4

Spin^c Cutting

In [4] Lerman describes the symplectic cutting construction for symplectic manifolds equipped with a Hamiltonian G -action. In [6] this construction is generalized to manifolds with other structures, including spin^c manifolds. However, the cutting of a spin^c-structure is incomplete in [6], since it only produces a spin^c principal bundle on the cut spaces $P_{cut} \rightarrow M_{cut}$, without constructing a map $P_{cut} \rightarrow SOF(M_{cut})$.

In this chapter, we describe the construction from section 6 in [6] and fill the necessary gaps.

From now on we will work with G -equivariant spin^c structures. This includes the non-equivariant case when G is taken to be the trivial group $\{e\}$.

4.1 The product of two spin^c structures

Note that the group $SO(m) \times SO(n)$ naturally embeds in $SO(n+m)$ as block matrices, and therefore it acts on $SO(n+m)$ from the left by left multiplication.

The proof of the following claim is straightforward.

Claim 4.1.1. *Let M and N be two oriented Riemannian manifolds of respective dimen-*

sions m and n . Then the map

$$\begin{aligned} (SOF(M) \times SOF(N)) \times_{SO(m) \times SO(n)} SO(n+m) &\rightarrow SOF(M \times N) \\ [(f, g), K] &\mapsto (f, g) \circ K \end{aligned}$$

is an isomorphism of principal $SO(n+m)$ -bundles.

Here, $f : \mathbb{R}^m \xrightarrow{\sim} T_a M$ and $g : \mathbb{R}^n \xrightarrow{\sim} T_b N$ are frames, and $K : \mathbb{R}^{m+n} \rightarrow \mathbb{R}^{m+n}$ is in $SO(m+n)$.

The above claim suggests a way to define the product of two spin^c manifolds (see also Lemma 6.10 from [6]). There is a natural group homomorphism $j : \text{Spin}(m) \times \text{Spin}(n) \rightarrow \text{Spin}(m+n)$, which is induced from the embeddings

$$\mathbb{R}^m \hookrightarrow \mathbb{R}^m \times \{0\} \subset \mathbb{R}^{m+n} \quad \text{and} \quad \mathbb{R}^n \hookrightarrow \{0\} \times \mathbb{R}^n \subset \mathbb{R}^{m+n} .$$

This gives rise to a homomorphism

$$j^c : \text{Spin}^c(m) \times \text{Spin}^c(n) \rightarrow \text{Spin}^c(m+n) \quad , \quad ([A, a], [N, b]) \mapsto [j(A, B), ab] ,$$

and therefore $\text{Spin}^c(m) \times \text{Spin}^c(n)$ acts from the left on $\text{Spin}^c(m+n)$ via j^c .

If a group G acts on two manifolds M and N , then it clearly acts on $M \times N$ by $g \cdot (m, n) = (g \cdot m, g \cdot n)$, and the above claim generalizes to this case as well.

Definition 4.1.1. Let G be a Lie group that acts on two oriented Riemannian manifolds M, N by orientation preserving isometries. Let $P_M \rightarrow SOF(M)$ and $P_N \rightarrow SOF(N)$ be G -equivariant spin^c structures on M and N . Then

$$P = (P_M \times P_N) \times_{\text{Spin}^c(m) \times \text{Spin}^c(n)} \text{Spin}^c(m+n) \rightarrow SOF(M \times N)$$

is a G -equivariant spin^c -structure on $M \times N$, called *the product* of the two given spin^c structures.

Remark 4.1.1. In the above setting, if L_M and L_N are the determinant line bundles of the spin^c structures on M and N , respectively, then the determinant line bundle of $P \rightarrow SOF(M \times N)$ is $L_M \boxtimes L_N$ (exterior tensor product). See Lemma 6.10 from [6] for details.

4.2 Restriction of a spin^c structure

In general, it is not clear how to restrict a spin^c -structure from a Riemannian oriented manifold to a submanifold. However, for our purposes, it suffices to work with co-oriented submanifolds of co-dimension 1.

The proof of the following claim is straightforward.

Claim 4.2.1. *Assume that the following data are given:*

1. M an oriented Riemannian manifold of dimension m .
2. G a Lie group that acts on M by orientation preserving isometries.
3. $Z \subset M$ a G -invariant co-oriented submanifold of co-dimension 1.
4. $P \rightarrow \text{SOF}(M)$ a G -equivariant spin^c -structure on M .

Define an injective map

$$i : \text{SOF}(Z) \rightarrow \text{SOF}(M) \quad , \quad i(f)(a_1, \dots, a_m) = f(a_1, \dots, a_{m-1}) + a_m \cdot v_p$$

where $f : \mathbb{R}^{m-1} \xrightarrow{\sim} T_p Z$ is a frame in $\text{SOF}(Z)$, and $v \in \Gamma(TM|_Z)$ is the vector field of positive unit vectors, orthogonal to TZ .

Then the pullback $P' = i^*(P) \rightarrow \text{SOF}(Z)$ is a G -equivariant spin^c -structure on Z , called the restriction of P to Z .

Remark 4.2.1.

1. This is the relevant commutative diagram for the claim:

$$\begin{array}{ccc}
 P' = i^*(P) & \longrightarrow & P \\
 \downarrow & & \downarrow \\
 \text{SOF}(Z) & \xrightarrow{i} & \text{SOF}(M) \\
 \downarrow & & \downarrow \\
 Z & \longrightarrow & M
 \end{array}$$

2. The principal action of $\text{Spin}^c(m-1)$ on P' is obtained using the natural inclusion $\text{Spin}^c(m-1) \hookrightarrow \text{Spin}^c(m)$.
3. The determinant line bundle of P' is the restriction to Z of the determinant line bundle of P .

4.3 Quotients of spin^c structures

We now discuss the process of taking quotients of a spin^c structure with respect to a group action. Since the basic cutting construction involves an S^1 -action, we will only deal with circle actions.

Assume that the following data are given:

1. An oriented Riemannian manifold Z of dimension n .
2. A free action $S^1 \curvearrowright Z$ by isometries.
3. $P \rightarrow \text{SOF}(Z)$ an S^1 -equivariant spin^c -structure on Z .

Denote by $\frac{\partial}{\partial \theta} \in \text{Lie}(S^1)$ an infinitesimal generator, by $(\frac{\partial}{\partial \theta})_Z \in \chi(Z)$ the corresponding vector field, and by $\pi : Z \rightarrow Z/S^1$ the quotient map. Also let $V = \pi^*(T(Z/S^1))$. This is an S^1 -equivariant vector bundle over Z .

$$\begin{array}{ccc}
 V = \pi^*(T(Z/S^1)) & \longrightarrow & T(Z/S^1) \\
 \downarrow & & \downarrow \\
 Z & \xrightarrow{\pi} & Z/S^1
 \end{array}$$

We have the following simple fact.

Lemma 4.3.1. *The map*

$$\left(\left(\frac{\partial}{\partial \theta} \right)_Z \right)^\perp \longrightarrow V \quad v \in T_p Z \longmapsto (p, \pi_* v) \in V_p$$

is an isomorphism of S^1 -equivariant vector bundles over Z .

Remark 4.3.1. Using this lemma, we can endow V with a Riemannian metric and orientation, and hence V becomes an oriented Riemannian vector bundle (of rank $n - 1$). We will think of V as a sub-bundle of TZ .

Also, if an orthonormal frame in V is chosen, then its image in $T(Z/S^1)$ is declared to be orthonormal. This endows Z/S^1 with an orientation and a Riemannian metric, and hence it makes sense to speak of $\text{SOF}(Z/S^1)$.

Now define a map $\eta: \text{SOF}(V) \rightarrow \text{SOF}(Z)$ in the following way. If $f: \mathbb{R}^{n-1} \xrightarrow{\cong} V_p$ is a frame, then $\eta(f): \mathbb{R}^n \rightarrow T_p Z$ will be given by $\eta(f)e_i = f(e_i)$ for $i = 1, \dots, n - 1$ and $\eta(f)e_n$ is a unit vector in the direction of $\left(\frac{\partial}{\partial \theta} \right)_{Z,p}$.

The following lemmas are used to get a spin^c structure on Z/S^1 . Their proofs are straightforward and left to the reader.

Lemma 4.3.2. *The pullback $\eta^*(P) \subset \text{SOF}(V) \times P$ is an S^1 -equivariant spin^c -structure on $\text{SOF}(V)$.*

(The S^1 -action on $\eta^(P)$ is induced from the S^1 -actions on $\text{SOF}(V)$ and P , and the right action of $\text{Spin}^c(n - 1)$ is induced by the natural inclusion $\text{Spin}^c(n - 1) \subset \text{Spin}^c(n)$).*

$$\begin{array}{ccc} \eta^*(P) & \longrightarrow & P \\ \downarrow & & \downarrow \\ \text{SOF}(V) & \xrightarrow{\eta} & \text{SOF}(Z) \\ \searrow & & \swarrow \\ & Z & \end{array}$$

Lemma 4.3.3. *Consider the S^1 -equivariant spin^c -structure $\eta^*(P) \rightarrow \text{SOF}(V) \rightarrow Z$. The quotient of each of the three components by the left S^1 action gives rise to a spin^c structure on Z/S^1 , called the quotient of the given spin^c -structure.*

$$\begin{array}{c} \bar{P} := \eta^*(P)/S^1 \\ \downarrow \\ \text{SOF}(Z/S^1) = \text{SOF}(V)/S^1 \\ \downarrow \\ Z/S^1 \end{array}$$

Remark 4.3.2. If L is the determinant line bundle of the given spin^c structure on Z , then the determinant line bundle of \bar{P} is L/S^1 .

4.4 Spin^c cutting

We are now in the position of describing the process of cutting a given S^1 -equivariant spin^c -structure on a manifold. Assume that the following data are given:

1. An oriented Riemannian manifold M of dimension m .
2. An action of S^1 on M by isometries.
3. A co-oriented submanifold $Z \subset M$ of co-dimension 1 that is S^1 -invariant. We also demand that S^1 acts freely on Z , and that $M \setminus Z$ is a disjoint union of two open pieces M_+ , M_- , such that positive (resp. negative) normal vectors point into M_+ (resp. M_-). Such submanifolds are called *reducible splitting hypersurfaces* (see Definitions 3.1 and 3.2 in [6]).
4. $P \rightarrow \text{SOF}(M)$ an S^1 -equivariant spin^c -structure on M .

We will use the following fact.

Claim 4.4.1. *There is an invariant (smooth) function $\Phi : M \rightarrow \mathbb{R}$, such that $\Phi^{-1}(0) = Z$, $\Phi^{-1}(0, \infty) = M_+$, $\Phi^{-1}(-\infty, 0) = M_-$ and 0 is a regular value of Φ .*

To prove this claim, first define Φ locally on a chart, use a partition of unity to get a globally well defined function on the whole manifold, and then average with respect to the group action to get S^1 -invariance.

This function Φ plays the role of a ‘moment map’ for the S^1 action. To define *the cut space* M_{cut}^+ , first introduce an S^1 -action on $M \times \mathbb{C}$

$$a \cdot (m, z) = (a \cdot m, a^{-1}z)$$

and then let $M_{cut}^+ = \{(m, z) | \Phi(m) = |z|^2\} / S^1$. The cut space M_{cut}^- is defined similarly, using the diagonal action on $M \times \mathbb{C}$

$$a \cdot (m, z) = (a \cdot m, a \cdot z)$$

and by setting $M_{cut}^- = \{(m, z) | \Phi(m) = -|z|^2\} / S^1$.

Remark 4.4.1. The orientation and the Riemannian metric on M (and on \mathbb{C}) descend to the cut spaces M_{cut}^\pm as follows. $M \times \mathbb{C}$ is naturally an oriented Riemannian manifold. Consider the map

$$\tilde{\Phi} : M \times \mathbb{C} \rightarrow \mathbb{R} \quad \tilde{\Phi}(m, z) = \Phi(m) - |z|^2$$

Zero is a regular value of $\tilde{\Phi}$, and therefore $\tilde{Z} = \tilde{\Phi}^{-1}(0)$ is a manifold. It inherits a metric and is co-oriented (hence oriented). Since S^1 acts freely on \tilde{Z} , the quotient $M_{cut}^+ = \tilde{Z} / S^1$ is an oriented Riemannian manifold (see Remark 4.3.1).

A similar procedure, using $\tilde{\Phi}(m, z) = \Phi(m) + |z|^2$, is carried out in order to get an orientation and a metric on M_{cut}^- .

We also have an S^1 action on the cut spaces (see Remark 4.4.2).

The purpose of this section is to describe how to get spin^c structures on M_{cut}^\pm from the given spin^c -structure on M . We start by constructing a spin^c -structure on M_{cut}^+ .

Step 1. Consider \mathbb{C} with its natural structure as an oriented Riemannian manifold, and let

$$P_{\mathbb{C}} = \mathbb{C} \times \text{Spin}^c(2) \longrightarrow \text{SOF}(\mathbb{C}) = \mathbb{C} \times \text{SO}(2) \longrightarrow \mathbb{C}$$

be the trivial spin^c -structure on \mathbb{C} . Turn it into an S^1 -equivariant spin^c -structure by letting S^1 act on $P_{\mathbb{C}}$:

$$e^{i\theta} \cdot (z, [a, b]) = (e^{-i\theta} z, [x_{-\theta/2} \cdot a, e^{i\theta/2} \cdot b]) \quad z \in \mathbb{C}, [a, b] \in \text{Spin}^c(2)$$

where $x_\theta = \cos \theta + \sin \theta \cdot e_1 e_2 \in \text{Spin}(2)$.

Here is a diagram for this structure.

$$\begin{array}{ccccc} S^1 \times P_{\mathbb{C}} & \longrightarrow & P_{\mathbb{C}} & \longleftarrow & P_{\mathbb{C}} \times \text{Spin}^c(2) \\ \downarrow & & \downarrow & & \downarrow \\ S^1 \times \text{SOF}(\mathbb{C}) & \longrightarrow & \text{SOF}(\mathbb{C}) & \longleftarrow & \text{SOF}(\mathbb{C}) \times \text{SO}(2) \\ \downarrow & & \downarrow & & \\ S^1 \times \mathbb{C} & \longrightarrow & \mathbb{C} & & \end{array}$$

Step 2. Taking the product of the spin^c structures P (on M) and $P_{\mathbb{C}}$ (on \mathbb{C}), we get an (S^1 equivariant) spin^c -structure $P_{M \times \mathbb{C}}$ on $M \times \mathbb{C}$ (see §4.1).

Step 3. It is easy to check that

$$\tilde{Z} = \{(m, z) \mid \Phi(m) = |z|^2\} \subset M \times \mathbb{C}$$

is an S^1 -invariant co-oriented submanifold of co-dimension one, and therefore we can restrict $P_{M \times \mathbb{C}}$ and get an S^1 -equivariant spin^c -structure $P_{\tilde{Z}}$ on \tilde{Z} (see §4.2).

Step 4. Since $P_{\tilde{Z}} \rightarrow \text{SOF}(\tilde{Z}) \rightarrow \tilde{Z}$ is an S^1 -equivariant spin^c structure, we can take the quotient by the S^1 -action to get a spin^c -structure P_{cut}^+ on $M_{cut}^+ = \tilde{Z}/S^1$ (see §4.3).

Remark 4.4.2. The spin^c -structure P_{cut}^+ can be turned into an S^1 -equivariant one. This is done by observing that we actually have two S^1 actions on $M \times \mathbb{C}$: the anti-diagonal action $a \cdot (m, z) = (a \cdot m, a^{-1} \cdot z)$ and the action on M : $a \cdot (m, z) = (a \cdot m, z)$. These actions commute with each other, and the action on M naturally descends to the cut space M_{cut}^+ and lifts to the spin^c -structure P_{cut}^+ .

Let us now describe briefly the analogous construction for M_{cut}^- .

Step 1. Define $P_{\mathbb{C}}$ as before, but with the action

$$e^{i\theta} \cdot (z, [a, b]) = (e^{i\theta} z, [x_{\theta/2} \cdot a, e^{i\theta/2} \cdot b])$$

Step 2. Define the spin^c -structure $P_{M \times \mathbb{C}}$ on $M \times \mathbb{C}$ as before.

Step 3. As before, replacing \tilde{Z} with $\{(m, z) | \Phi(m) = -|z|^2\} \subset M \times \mathbb{C}$.

Step 4. Repeat as before to get a spin^c -structure P_{cut}^- on M_{cut}^- .

Remark 4.4.3. In step 1 we defined a spin^c -structure on \mathbb{C} . The corresponding determinant line bundle is the trivial line bundle $\mathbb{L}_{\mathbb{C}} = \mathbb{C} \times \mathbb{C}$ over \mathbb{C} (with projection $(z, b) \mapsto z$).

The S^1 action on $\mathbb{L}_{\mathbb{C}}$ is given by

$$a \cdot (z, b) = \begin{cases} (a^{-1} \cdot z, a \cdot b) & \text{for } P_{cut}^+ \\ (a \cdot z, a \cdot b) & \text{for } P_{cut}^- \end{cases}$$

If \mathbb{L} is the determinant line bundle of the given spin^c -structure on M , then the determinant line bundle on M_{cut}^{\pm} is given by

$$\mathbb{L}_{cut}^{\pm} = [(\mathbb{L} \boxtimes \mathbb{L}_{\mathbb{C}}) |_{\tilde{Z}}] / S^1$$

where we divide by the diagonal action of S^1 on $\mathbb{L} \times \mathbb{L}_{\mathbb{C}}$. This is an S^1 -equivariant complex line bundle (with respect to the action on M).

Chapter 5

The Generalized Kostant Formula for Isolated Fixed Points

Assume that the following data are given:

1. An oriented compact Riemannian manifold M of dimension $2m$.
2. $T = \mathbb{T}^n$ an n -dimensional torus that acts on M by isometries.
3. $P \rightarrow SOF(M)$ a T -equivariant spin^c -structure, with determinant line bundle \mathbb{L} .
4. A $U(1)$ and T -invariant connection on $P_1 = P/Spin(2m)$.

As we saw in §3.5, this data determines a complex virtual representation $Q(M) = \ker(D^+) - \text{coker}(D^+)$ of T . Denote by $\chi: T \rightarrow \mathbb{C}$ its character.

Lemma 5.0.1. *Let $x \in M^T$ be a fixed point, and choose a T -invariant complex structure $J: T_x M \rightarrow T_x M$. Denote by $\alpha_1, \dots, \alpha_m \in \mathfrak{t}^* = \text{Lie}(T)^*$ the weights of the action $T \curvearrowright T_x M$, and by μ the weight of $T \curvearrowright \mathbb{L}_x$. Then $\frac{1}{2} \left(\mu - \sum_{j=1}^m \alpha_j \right)$ is in the weight lattice of T .*

Proof. Decompose $T_x M = L_1 \oplus \cdots \oplus L_m$, where each L_j is a 1-dimensional T -invariant complex subspace of $T_x M$, on which T acts with weight α_j . Fix a point $p \in P_x$.

For each $z \in T$, there is a unique element $[A_z, w_z] \in Spin^c(2m)$ such that $z \cdot p = p \cdot [A_z, w_z]$. This gives a homomorphism

$$\eta: T \rightarrow Spin^c(2m) \quad , \quad z \mapsto [A_z, w_z]$$

(note that A_z and w_z are defined only up to sign, but the element $[A_z, w_z]$ is well defined).

Choose a basis $\{e_j\} \subset T_x M$ (over \mathbb{C}) with $e_j \in L_j$ for all $1 \leq j \leq m$. With respect to this basis, each element $z \in T$ acts on $T_x M$ through the matrix

$$A'_z = \begin{pmatrix} z^{\alpha_1} & & & 0 \\ & z^{\alpha_2} & & \\ & & \ddots & \\ 0 & & & z^{\alpha_m} \end{pmatrix} \in U(m) \subset SO(2m) .$$

This enables us to define another homomorphism

$$\eta': T \rightarrow SO(2m) \times S^1 \quad , \quad z \mapsto (A'_z, z^\mu) .$$

It is not hard to see that the relation $z \cdot p = p \cdot [A_z, w_z]$ (for all $z \in T$) will imply the commutativity of the following diagram.

$$\begin{array}{ccc} & Spin^c(2m) & \\ & \nearrow \eta & \downarrow \\ T & \xrightarrow{\eta'} & SO(2m) \times S^1 \end{array}$$

(The vertical map is the double cover taking $[A, z] \in Spin^c(2m)$ to $(\lambda(A), z^2)$). For any $z = exp(\theta) \in T$, $\theta \in \mathfrak{t}$, we have

$$\lambda(A_z) = A'_z \quad \Rightarrow \quad A_z = \prod_{j=1}^m \left[\cos\left(\frac{\alpha_j(\theta)}{2}\right) + \sin\left(\frac{\alpha_j(\theta)}{2}\right) e_j J(e_j) \right] \in Spin(2m)$$

(where the spin group is thought of as sitting inside the Clifford algebra)

and

$$w_z^2 = z^\mu \quad \Rightarrow \quad w_z = z^{\mu/2} .$$

Note that

$$T_{Spin^c(2m)} = \left\{ \left[\prod_{j=1}^m (\cos t_j + \sin t_j \cdot e_j J(e_j)), u \right] : t_j \in \mathbb{R} , u \in S^1 \right\} \subset Spin^c(2m)$$

is a maximal torus, and that in fact η is a map from T to $T_{Spin^c(2m)}$.

Now define another map

$$\psi : T_{Spin^c(2m)} \rightarrow S^1 \quad , \quad \left[\prod_{j=1}^m (\cos t_j + \sin t_j \cdot e_j J(e_j)), u \right] \mapsto u \cdot e^{-i \sum_j t_j}$$

By composing η and ψ we get a well defined map $\psi \circ \eta : T \rightarrow S^1$ which is given by

$$exp(\theta) \mapsto (e^{i\theta})^{\frac{1}{2}(\mu - \sum_j \alpha_j)(\theta)}$$

and therefore $\frac{1}{2}(\mu - \sum_j \alpha_j)$ must be a weight of T . □

Remark 5.0.4. The idea in the above proof is simple. To show that $\beta = \frac{1}{2}(\mu - \sum_j \alpha_j)$ is a weight, we want to construct a 1-dimensional complex representation of T with weight β . The map η is a natural homomorphism $T \rightarrow Spin^c(2m)$. The map ψ is nothing but the action of a maximal torus of $Spin^c(2m)$ on the lowest weight space of the spin representation Δ_{2m}^+ (see Proposition 3.3.2, and Lemma 12.12 in [7]). Finally, $\psi \circ \eta : T \rightarrow S^1$ is the required representation.

The following is proposition 11.3 from [7].

Proposition 5.0.1. *Assume that the fixed points M^T of the action on M are isolated.*

For each $p \in M^T$, choose a complex structure on $T_p M$, and denote by

1. $\alpha_{1,p}, \dots, \alpha_{m,p} \in \mathfrak{t}^*$ the weights of the action of T on $T_p M$.
2. μ_p the weight of the action of T on \mathbb{L}_p .
3. $(-1)^p$ will be $+1$ if the orientation coming from the choice of the complex structure on $T_p M$ coincides with the orientation of M , and -1 otherwise.

Then the character $\chi: T \rightarrow \mathbb{C}$ of $Q(M)$ is given by

$$\chi(\lambda) = \sum_{p \in M^G} \nu_p(\lambda) \quad \nu_p(\lambda) = (-1)^p \cdot \lambda^{\mu_p/2} \prod_{j=1}^m \frac{\lambda^{-\alpha_{j,p}/2} - \lambda^{\alpha_{j,p}/2}}{(1 - \lambda^{\alpha_{j,p}})(1 - \lambda^{-\alpha_{j,p}})}$$

where $\lambda^\beta: T \rightarrow S^1$ is the representation that corresponds to the weight $\beta \in \mathfrak{t}^*$.

Remark 5.0.5.

1. Although $\pm\alpha_{j,p}/2$ may not be in the weight lattice of T , the expression $\nu_p(\lambda)$, can be equivalently written as

$$(-1)^p \cdot \lambda^{(\mu_p - \sum_j \alpha_{j,p})/2} \prod_{j=1}^m \frac{1 - \lambda^{\alpha_{j,p}}}{(1 - \lambda^{\alpha_{j,p}})(1 - \lambda^{-\alpha_{j,p}})}.$$

By Lemma 5.0.1, $(\mu_p - \sum_j \alpha_{j,p})/2$ is a weight, so $\nu_p(\lambda)$ is well defined.

2. Since the fixed points of the action $T \curvearrowright M$ are isolated, all the $\alpha_{j,p}$'s are nonzero. This follows easily from Theorem B.26 in [2].

Now we present the generalized Kostant formula for spin^c quantization.

Assume that the fixed points of $T \curvearrowright M$ are isolated, choose a complex structure on $T_p M$ for each $p \in M^T$, and use the notation of Proposition 5.0.1. By the above remark, we can find a *polarizing vector* $\xi \in \mathfrak{t}$ such that $\alpha_{j,p}(\xi) \neq 0$ for all j, p . We can choose our complex structures on $T_p M$ such that $\alpha_{j,p}(\xi) \in i\mathbb{R}^+$ for all j, p .

For each weight $\beta \in \mathfrak{t}^*$ denote by $\#(\beta, Q(M))$ the multiplicity of this weight in $Q(M)$. Also, for $p \in M^T$ define the partition function $\bar{N}_p: \mathfrak{t}^* \rightarrow \mathbb{Z}^+$ by setting:

$$\bar{N}_p(\beta) = \left| \left\{ (k_1, \dots, k_m) \in \left(\mathbb{Z} + \frac{1}{2} \right)^m : \beta + \sum_{j=1}^m k_j \alpha_{j,p} = 0 \quad , \quad k_j > 0 \right\} \right|$$

The right hand side is always finite since our weights are polarized.

Theorem 5.0.1 (Kostant formula). *For any weight $\beta \in \mathfrak{t}^*$ of T , we have*

$$\#(\beta, Q(M)) = \sum_{p \in M^T} (-1)^p \cdot \bar{N}_p \left(\beta - \frac{1}{2} \mu_p \right)$$

Proof. For $p \in M^T$ and $\lambda \in T$, set $\alpha_j = \alpha_{j,p}$ and $\mu = \mu_p$. From Proposition 5.0.1 we then get

$$\nu_p(\lambda) = (-1)^p \cdot \lambda^{\mu/2} \prod_{j=1}^m \frac{\lambda^{-\alpha_j/2}(1 - \lambda^{\alpha_j})}{(1 - \lambda^{\alpha_j})(1 - \lambda^{-\alpha_j})} = (-1)^p \cdot \lambda^{\frac{1}{2}(\mu - \sum_j \alpha_j)} \prod_{j=1}^m \frac{1}{1 - \lambda^{-\alpha_j}}$$

Note that we have

$$\prod_{j=1}^m \frac{1}{1 - \lambda^{-\alpha_j}} = \sum_{\beta} N_p(\beta) \cdot \lambda^{\beta}$$

Where the sum is taken over all weights $\beta \in \mathfrak{t}^*$ in the weight lattice ℓ^* of T and $N_p(\beta)$ is the number of *non-negative* integer solutions $(k_1, \dots, k_m) \in (\mathbb{Z}_+)^m$ to

$$\beta + \sum_{j=1}^m k_j \alpha_j = 0$$

(see formula 5 in [5]). Hence,

$$\nu_p(\lambda) = (-1)^p \cdot \sum_{\beta \in \ell^*} N_p(\beta) \cdot \lambda^{\beta + \frac{1}{2}(\mu - \sum_j \alpha_j)}$$

By Lemma 5.0.1, $\frac{1}{2}(\mu - \sum_j \alpha_j) \in \ell^*$ (i.e., it is a weight), so by change of variable $\beta \mapsto \beta - \frac{1}{2}(\mu - \sum_j \alpha_j)$ we get

$$\nu_p(\lambda) = (-1)^p \cdot \sum_{\beta \in \ell^*} N_p \left(\beta - \frac{1}{2}\mu + \frac{1}{2} \sum_j \alpha_j \right) \cdot \lambda^{\beta}$$

By definition, $N_p \left(\beta - \frac{1}{2}\mu + \frac{1}{2} \sum_j \alpha_j \right)$ is the number of non-negative integer solutions for the equation

$$\beta - \frac{1}{2}\mu + \frac{1}{2} \sum_j \alpha_j + \sum_j k_j \alpha_j = 0$$

or, equivalently, to

$$\beta - \frac{1}{2}\mu + \sum_j \left(k_j + \frac{1}{2} \right) \alpha_j = 0$$

Using the definition of \overline{N}_p (see above) we conclude that

$$N_p \left(\beta - \frac{1}{2}\mu + \frac{1}{2} \sum_j \alpha_j \right) = \overline{N}_p \left(\beta - \frac{1}{2}\mu \right)$$

and then

$$\nu_p(\lambda) = (-1)^p \cdot \sum_{\beta \in \ell^*} \bar{N}_p \left(\beta - \frac{1}{2}\mu \right) \lambda^\beta$$

This means that the formula for the character can be written as

$$\chi(\lambda) = \sum_{\beta \in \ell^*} \left[\sum_{p \in M^T} (-1)^p \cdot \bar{N}_p \left(\beta - \frac{1}{2}\mu \right) \right] \lambda^\beta$$

and the multiplicity of β in $Q(M)$ is given by

$$\#(\beta, Q(M)) = \sum_{p \in M^T} (-1)^p \cdot \bar{N}_p \left(\beta - \frac{1}{2}\mu \right)$$

as desired. □

Chapter 6

The Generalized Kostant Formula for Non-Isolated Fixed Points

6.1 Equivariant characteristic classes

Let an *abelian* Lie group G (with Lie algebra \mathfrak{g}) act *trivially* on a smooth manifold X . We now define the equivariant cohomology (with generalized coefficients) and equivariant characteristic classes for this special case. For the more general case, see [9] or Appendix C in [2].

Definition 6.1.1. A real-valued function α is called an *almost everywhere analytic function* (a.e.a) if

1. Its domain is of the form $\mathfrak{g} \setminus P$, and $P \subset \mathfrak{g}$ is a closed set of measure zero.
2. It is analytic on $\mathfrak{g} \setminus P$.

Denote by $C^\#(\mathfrak{g})$ the space of all equivalence classes of a.e.a functions on \mathfrak{g} (two such functions are equivalent if they coincide outside a set of measure zero).

Let $\mathcal{A}_G^\#(X) = C^\#(\mathfrak{g}) \otimes \Omega^\bullet(X; \mathbb{C})$ be the space of all a.e.a functions $\mathfrak{g} \rightarrow \Omega^\bullet(X; \mathbb{C})$, where $\Omega^\bullet(X; \mathbb{C})$ is the (ordinary) de Rham complex of X with complex coefficients.

Define a differential (recall that G is abelian and the action is trivial)

$$d_{\mathfrak{g}} : \mathcal{A}_G^{\#}(X) \rightarrow \mathcal{A}_G^{\#}(X) \quad (d_{\mathfrak{g}}\alpha)(u) = d(\alpha(u))$$

and the G -equivariant (de Rham) cohomology of X

$$H_G^{\#}(X) = \frac{Ker(d_{\mathfrak{g}})}{Im(d_{\mathfrak{g}})} .$$

Note that $H_G^{\#}(X)$ is isomorphic to the space $C^{\#}(\mathfrak{g}) \otimes H^{\bullet}(X; \mathbb{C})$ of a.e.a functions $\mathfrak{g} \rightarrow H^{\bullet}(X; \mathbb{C})$. Equivariant characteristic classes will be elements of the ring $H_G^{\#}(X)$.

If X is compact and oriented, then equivariant cohomology classes can be integrated over X . For any class $[\alpha] \in H_G^{\#}(X)$ and u in the domain of α , let

$$\left(\int_X [\alpha] \right) (u) = \int_X (\alpha(u))$$

and thus $\int_X [\alpha]$ is an element of $C^{\#}(\mathfrak{g}) \otimes \mathbb{C}$.

Assume now that both X and G are connected, and let $\pi : L \rightarrow X$ be a complex line bundle over X . Assume that G acts on the fibers of the bundle with weight $\mu \in \mathfrak{g}^*$, i.e., $\exp(u) \cdot y = e^{i\mu(u)} \cdot y$ for all $u \in \mathfrak{g}$ and $y \in L$ (so the action on the base space is still trivial). Denote by $c_1(L) = [\omega] \in H^2(X)$ the (ordinary) first Chern class of the line bundle. Here $\omega \in \Omega^2(X)$ is a real two-form. Then the *first equivariant Chern class* of the equivariant line bundle $L \rightarrow X$ is defined to be $[\omega + \mu] \in H_G^{\#}(X)$. We will denote this class by $\tilde{c}_1(L)$.

Now assume that $E \rightarrow X$ is a G -equivariant complex vector bundle of complex rank k (where G acts trivially on X), that splits as a sum of k equivariant complex line bundles $E = L_1 \oplus \cdots \oplus L_k$ (one can avoid this assumption by using *the (equivariant) splitting principle*). Let $\tilde{c}_1(L_1) = [\omega_1 + \mu_1], \dots, \tilde{c}_1(L_k) = [\omega_k + \mu_k]$ be the equivariant first Chern classes of these line bundles, and define *the equivariant Euler class* of E by

$$\tilde{E}u(E) = \prod_{j=1}^k \tilde{c}_1(L_j) = \left[\prod_{j=1}^k (\omega_j + \mu_j) \right] \in H_G^{\#}(X).$$

We will also need the equivariant A -roof class, which we will denote by $\tilde{A}(E)$. To define this class, consider the following meromorphic function

$$f(z) = \frac{z}{e^{z/2} - e^{-z/2}} = \frac{z/2}{\sinh(z/2)} \quad f(0) = 1 .$$

Its domain is $D = \mathbb{C} \setminus \{\pm 2\pi i, \pm 4\pi i, \dots\}$. Define, for each $1 \leq j \leq k$,

$$f(\tilde{c}_1(L_j))(u) = f(c_1(L_j) + \mu_j(u)) = \sum_{n=1}^{\infty} \frac{f^{(n)}(\mu_j(u))}{n!} \cdot (c_1(L_j))^n$$

whenever $\mu_j(u) \in D$ for all $1 \leq j \leq k$, and also

$$\tilde{A}(E) = \prod_{j=1}^k f(\tilde{c}_1(L_j)).$$

Also note that the quotient

$$\frac{\tilde{A}(E)}{\tilde{E}u(E)}$$

can be defined using the same procedure, replacing $f(z)$ with $\frac{1}{2 \sinh(z/2)}$. If all the μ_j 's are nonzero, then

$$\frac{\tilde{A}(E)}{\tilde{E}u(E)} \in H_G^\#(X).$$

6.2 The Kostant formula

Assume that the following data are given:

1. An oriented compact Riemannian manifold M of dimension $2m$.
2. A circle action $S^1 \curvearrowright M$ by isometries.
3. An S^1 -equivariant spin^c -structure $P \rightarrow \text{SOF}(M)$, with determinant line bundle \mathbb{L} .
4. A $U(1)$ and S^1 -invariant connection on $P_1 = P/\text{Spin}(2m) \rightarrow M$.

In this section we present a formula for the character $\chi: S^1 \rightarrow \mathbb{C}$ of the virtual representation $Q(M)$ determined by the above data (see §3.5). We *do not* assume, however, that the fixed points are isolated.

We use the following conventions and notation.

- M^{S^1} is the fixed points set.
- For each connected component $F \subset M^{S^1}$, let NF denote the normal bundle to $TF \subset TM$. The bundles NF and TF are S^1 -equivariant real vector bundles of even rank, with trivial fixed subspace, and therefore are equivariantly isomorphic to complex vector bundles. Choose an equivariant complex structure on the fibers of TF and NF , and denote the rank of NF as a complex vector bundle by $m(F)$.
- The complex structures on NF and TF induce an orientation on those bundles. Let $(-1)^F$ be $+1$ if the orientation of F followed by that of NF is the given orientation on M , and -1 otherwise.

With respect to the above data, choices and notation, we have

Proposition 6.2.1. *For all $u \in \mathfrak{g} = \text{Lie}(S^1)$ such that the right hand side is defined,*

$$\chi(\exp(u)) = \sum_{F \subset M^{S^1}} (-1)^F \cdot (-1)^{m(F)} \cdot \int_F e^{\frac{1}{2}\tilde{c}_1(\mathbb{L}|_F)} \cdot \tilde{A}(TF) \cdot \frac{\tilde{A}(NF)}{\tilde{E}u(NF)}$$

where the sum is taken over the connected components of M^{S^1} .

This formula is derived from the Atiyah-Segal-Singer index theorem (see [10]). For some details, see p.547 in [6].

Without loss of generality we may assume, according to the splitting principle, that the normal bundle splits as a direct sum of (equivariant) complex line bundles

$$NF = L_1^F \oplus \cdots \oplus L_{m(F)}^F \quad .$$

For each fixed component $F \subset M^{S^1}$, denote by $\{\alpha_{j,F}\}$ the weights of the action of S^1 on $\{L_j^F\}$. As in the previous section, all the $\alpha_{j,F}$'s are nonzero, and we can polarize them, i.e., we can choose our complex structure on NF in such a way that $\alpha_{j,F}(\xi) > 0$

for some fixed $\xi \in \mathfrak{g}$ and for all j 's and F 's. Also denote by μ_F the weight of the action of S^1 on $\mathbb{L}|_F$.

For each $\beta \in \mathfrak{g}^* = \text{Lie}(S^1)^*$, define the following set (which is finite, since our weights are polarized)

$$\mathcal{S}_\beta = \left\{ (k_1, \dots, k_{m(F)}) \in \left(\mathbb{Z} + \frac{1}{2} \right)^{m(F)} : \beta + \sum_{j=1}^{m(F)} k_j \alpha_{j,F} = 0, \quad k_j > 0 \right\}$$

and for each tuple $k = (k_1, \dots, k_{m(F)})$, let

$$\bar{p}_{k,F} = (-1)^{m(F)} \int_F e^{\frac{1}{2}(c_1(\mathbb{L}|_F) - \sum_j c_1(L_j^F))} \cdot \tilde{A}(TF) \cdot e^{-\sum_j k_j c_1(L_j^F)} \quad .$$

Now define

$$\bar{N}_F(\beta) = \sum_{k \in \mathcal{S}_\beta} \bar{p}_{k,F} \quad .$$

With this notation, the Kostant formula in this case of nonisolated fixed points becomes identical to the formula for isolated fixed points (from §5).

Theorem 6.2.1. *For each weight $\beta \in \mathfrak{g}^* = \text{Lie}(S^1)^*$, the multiplicity of β in $Q(M)$ is given by*

$$\#(\beta, Q(M)) = \sum_{F \subset M^{S^1}} (-1)^F \cdot \bar{N}_F \left(\beta - \frac{1}{2} \mu_F \right),$$

where the sum is taken over the connected components of M^{S^1} .

Proof. For a fixed connected component $F \subset M^{S^1}$, omit the F in $\alpha_{j,F}$, μ_F and L_j^F , and

compute

$$\begin{aligned}
& \int_F e^{\frac{1}{2}\tilde{c}_1(\mathbb{L}|_F)} \cdot \tilde{A}(TF) \cdot \frac{\tilde{A}(NF)}{\tilde{E}u(NF)} = \\
& = \int_F e^{\frac{1}{2}c_1(\mathbb{L}|_F) + \frac{1}{2}\mu} \cdot \tilde{A}(TF) \cdot \prod_{j=1}^{m(F)} \frac{1}{e^{[c_1(L_j) + \alpha_j]/2} - e^{-[c_1(L_j) + \alpha_j]/2}} = \\
& = e^{\frac{1}{2}\mu} \cdot \int_F e^{\frac{1}{2}c_1(\mathbb{L}|_F)} \cdot \tilde{A}(TF) \cdot \prod_{j=1}^{m(F)} \frac{e^{-[c_1(L_j) + \alpha_j]/2}}{1 - e^{-[c_1(L_j) + \alpha_j]}} = \\
& = e^{[\mu - \sum_j \alpha_j]/2} \cdot \int_F e^{[c_1(\mathbb{L}|_F) - \sum_j c_1(L_j)]/2} \cdot \tilde{A}(TF) \cdot \prod_{j=1}^{m(F)} \frac{1}{1 - e^{-[c_1(L_j) + \alpha_j]}}
\end{aligned}$$

Using the geometric series

$$\frac{1}{1-z} = \sum_{l=0}^{\infty} z^l$$

and the notation $z = \exp(u)$ we get, for each j , and for each $u \in \mathfrak{g}$ such that the series converges,

$$\frac{1}{1 - e^{-[c_1(L_j) + \alpha_j(u)]}} = \sum_{l=0}^{\infty} e^{-l \cdot [c_1(L_j) + \alpha_j(u)]} = \sum_{l=0}^{\infty} e^{-l \cdot c_1(L_j)} z^{-l \cdot \alpha_j}$$

(where $z^{-l \cdot \alpha_j}$ is the representation of S^1 that corresponds to the weight $-l \cdot \alpha_j \in \ell^* \subset \mathfrak{g}^*$)

and thus

$$\prod_{j=1}^{m(F)} \frac{1}{1 - e^{-[c_1(L_j) + \alpha_j(u)]}} = \sum_{l \in \ell^*} \left[\sum_{l + \sum_j k_j \alpha_j = 0} e^{-\sum_j k_j c_1(L_j)} \right] z^l$$

The formula that we get for the character is

$$\begin{aligned}
\chi(z) &= \sum_{F \subset M^{S^1}} (-1)^F \cdot (-1)^{m(F)} \cdot e^{\frac{1}{2}[\mu_F(u) - \sum_j \alpha_{j,F}(u)]} \\
&\cdot \int_F e^{\frac{1}{2}[c_1(\mathbb{L}|_F) - \sum_j c_1(L_j^F)]} \cdot \tilde{A}(TF) \cdot \sum_{l \in \ell^*} \left[\sum_{l + \sum_j k_j \alpha_{j,F} = 0} e^{-\sum_j k_j c_1(L_{j,F})} \right] z^l = \\
&= \sum_{l \in \ell^*} \sum_{F \subset M^{S^1}} \sum_{l + \sum_j k_j \alpha_{j,F} = 0} (-1)^F \cdot \bar{p}_{k,F} \cdot z^{l + \frac{1}{2}(\mu_F - \sum_j \alpha_{j,F})}
\end{aligned}$$

Lemma 5.0.1 implies that $\frac{1}{2}(\mu_F - \sum_j \alpha_{j,F})$ is a weight of S^1 (so the previous formula is well defined), hence we can make a change of variables $\beta = l + \frac{1}{2}\mu_F - \frac{1}{2}\sum_j \alpha_{j,F}$ and get

$$\chi(z) = \sum_{\beta \in \ell^*} \left[\sum_{F \subset M^{S^1}} \sum_{k \in \mathcal{S}_{\beta - \frac{1}{2}\mu_F}} (-1)^F \cdot \bar{p}_{k,F} \right] z^\beta = \sum_{\beta \in \ell^*} \left[\sum_{F \subset M^{S^1}} (-1)^F \cdot \bar{N}_F \left(\beta - \frac{1}{2}\mu_F \right) \right] z^\beta$$

From this we conclude that the multiplicity of $\beta \in \ell^* \subset \mathfrak{g}^*$ in $Q(M)$ is given by

$$\#(\beta, Q(M)) = \sum_{F \subset M^{S^1}} (-1)^F \cdot \bar{N}_F \left(\beta - \frac{1}{2}\mu_F \right)$$

as desired (the sum is taken over the connected components of the fixed point set M^{S^1}). \square

6.3 The case $m(F) = 1$

To prove the additivity of spin^c quantization under cutting, we will need the terms of the Kostant formula for non-isolated fixed points in the special case where $m(F) = 1$, i.e., when the normal bundle to the fixed components has complex dimension 1. Therefore, assume that we are given the same data as in §6.2, and also that

- Each fixed component $F \subset M^{S^1}$ is of real codimension 2 in M , i.e., the normal bundle $NF = TM/TF$ is of real dimension 2.

For a fixed component F , we adopt all the notation from §6.2. Since $m(F)$ is assumed to be 1, we have

$$NF = L_1^F$$

and only one weight

$$\alpha_{1,F} = \alpha_F.$$

For each $\beta \in \mathfrak{g}^*$, the corresponding set \mathcal{S}_β becomes

$$\mathcal{S}_\beta = \left\{ k \in \mathbb{Z} + \frac{1}{2} \quad : \quad \beta + k \cdot \alpha_F = 0 \quad , \quad k > 0 \right\}$$

which is either empty or contains only one element. The expression for $\bar{p}_{k,F}$ also simplifies

to

$$\bar{p}_{k,F} = - \int_F e^{[c_1(\mathbb{L}|_F) - c_1(NF)]/2} \cdot \tilde{A}(TF) \cdot e^{-k \cdot c_1(NF)} ,$$

and this implies that

$$\bar{N}_F \left(\beta - \frac{1}{2} \mu_F \right) = \begin{cases} 0 & \text{if } \mathcal{S}_{\beta - \frac{1}{2} \mu_F} = \phi \\ \bar{p}_{k,F} & \text{if } \mathcal{S}_{\beta - \frac{1}{2} \mu_F} = \{k\} \end{cases} .$$

Chapter 7

Additivity under Cutting

In this chapter we prove our main result, namely, the additivity of spin^c quantization under the cutting construction described in §4.4 .

Our setting is as follows:

1. A compact oriented connected Riemannian manifold M of dimension $2m$.
2. An action of S^1 on M by isometries.
3. An S^1 -equivariant spin^c -structure $P \rightarrow SOF(M) \rightarrow M$.
4. A co-oriented splitting hypersurface $Z \subset M$ on which S^1 acts freely.

After choosing a $U(1)$ and S^1 -invariant connection on $P_1 = P/Spin(2m)$, we can construct a Dirac operator D^+ , whose index $Q(M)$ is independent of the connection. We call $Q(M)$ *the spin^c quantization of M* (see §3.5).

We can now perform the cutting construction from §4.4 to obtain two other manifolds M_{cut}^\pm (the *cut spaces*). Those cut spaces are also compact oriented Riemannian manifolds of dimension $2m$, endowed with a circle action and with S^1 -equivariant spin^c structures P_{cut}^\pm . Thus, we can quantize them (after choosing a suitable connection), and obtain two virtual representations $Q(M_{cut}^\pm)$.

Theorem 7.0.1. *As virtual representations of S^1 , we have*

$$Q(M) = Q(M_{cut}^+) \oplus Q(M_{cut}^-)$$

We will need a few preliminary lemmas for the proof of the theorem. Those are similar to Proposition 6.1 from [6], where a few gaps were found.

7.1 First lemma - The normal bundle

Recall the construction of M_{cut}^\pm from §4.4.

- Choose an S^1 -invariant smooth function $\phi: M \rightarrow \mathbb{R}$ such that $\phi^{-1}(0) = Z$, $\phi^{-1}(0, \infty) = M_+$, $\phi^{-1}(-\infty, 0) = M_-$, and 0 is a regular value of ϕ .
- Define $\tilde{Z}^\pm = \{(m, z) \mid \phi(m) = \pm|z|^2\} \subset M \times \mathbb{C}$, and let S^1 act on \tilde{Z}^\pm by $a \cdot (m, z) = (a \cdot m, a^{\mp 1} \cdot z)$.
- Finally, define $M_{cut}^\pm = \tilde{Z}^\pm / S^1$.

Remark 7.1.1. Note that we have S^1 -equivariant embeddings

$$Z \rightarrow \tilde{Z}^\pm, m \mapsto (m, 0) \quad \text{and} \quad Z/S^1 \rightarrow M_{cut}^\pm, [m] \mapsto [m, 0]$$

and therefore we can think of Z and Z/S^1 as submanifolds of \tilde{Z}^\pm and M_{cut}^\pm , respectively.

Lemma 7.1.1.

1. *The maps*

$$\eta: T(\tilde{Z}^\pm)|_Z \rightarrow Z \times \mathbb{C} \quad \eta: (v, w) \in T_{(m,0)}\tilde{Z}^\pm \mapsto (m, w)$$

give rise to short exact sequences

$$0 \longrightarrow TZ \longrightarrow T\tilde{Z}^\pm|_Z \xrightarrow{\eta} Z \times \mathbb{C} \longrightarrow 0$$

of S^1 -equivariant vector bundles (with respect to both the diagonal (anti-diagonal) action and the action on M) over Z . The action on $Z \times \mathbb{C}$ is taken to be

$$a \cdot (m, z) = (a \cdot m, a^{\mp 1} \cdot z) .$$

2. The short exact sequences above descend to the following short exact sequences

$$0 \longrightarrow T(Z/S^1) \longrightarrow T(M_{cut}^{\pm})|_{Z/S^1} \longrightarrow Z \times_{S^1} \mathbb{C} \longrightarrow 0$$

of equivariant vector bundles over Z/S^1 . The S^1 action on $Z \times_{S^1} \mathbb{C}$ is induced from the action on Z .

Proof.

1. The S^1 -equivariant embedding $Z \rightarrow \tilde{Z}^{\pm}$ gives rise to an injective map $TZ \rightarrow T\tilde{Z}^{\pm}$, which is an S^1 -equivariant map of vector bundles over Z . The map η is onto, since for any $(m, w) \in Z \times \mathbb{C}$ we have $\eta(0, w) = (m, w)$, and it is equivariant since for $(v, w) \in T_{(m,0)}\tilde{Z}^{\pm}$, $m \in Z$ we have

$$\eta(a \cdot (v, w)) = \eta(a \cdot v, a^{\mp} \cdot w) = (a \cdot m, a^{\mp} \cdot w) = a \cdot (m, z)$$

(and similarly for the action on M).

To prove $\ker(\eta) = TZ$, note that the definitions of ϕ and \tilde{Z} imply that

$$T\tilde{Z}^{\pm} = \left\{ (v, w) \in T_{(m,z)}M \times \mathbb{C} : d\phi_m(v) = \pm (z \cdot \bar{w} + \bar{z} \cdot w) , (m, z) \in \tilde{Z}^{\pm} \right\}$$

$$TZ = \{v \in T_m M : d\phi_m(v) = 0 , m \in Z\}$$

so $(v, w) \in T_{(m,0)}\tilde{Z}^{\pm}$ satisfies $\eta(v, w) = (m, 0)$ if and only if

$$w = 0 \quad \text{and} \quad d\phi_m(v) = 0 \Leftrightarrow v = (v, 0) \in T_m Z \subset T_{(m,0)}\tilde{Z}^{\pm}$$

and hence $\ker(\eta) = TZ$ and the sequence is exact.

2. is a direct consequence of (1).

□

Let $N^\pm \rightarrow Z$ be the normal bundle to Z in \tilde{Z}^\pm , and $\overline{N}^\pm \rightarrow Z/S^1$ be the normal bundle to Z/S^1 in M_{cut}^\pm . The above lemma implies:

Corollary 7.1.1. *The short exact sequences of Lemma 7.1.1 induce isomorphisms*

$$N^\pm \xrightarrow{\simeq} Z \times \mathbb{C} \quad \overline{N}^\pm \xrightarrow{\simeq} Z \times_{S^1} \mathbb{C}$$

of equivariant vector bundle, and hence an orientation on the fibers of the bundles \overline{N}^\pm (coming from the complex orientation on \mathbb{C}).

Remark 7.1.2. Note that the map

$$\overline{N}^+ = Z \times_{S^1} \mathbb{C} \longrightarrow \overline{N}^- = Z \times_{S^1} \mathbb{C} \quad , \quad [z, a] \mapsto [z, \bar{a}]$$

is an S^1 -equivariant orientation-reversing bundle isomorphism.

Claim 7.1.1. *The natural orientation on $Z/S^1 \subset M_{cut}^\pm$, coming from the reduction process, followed by the orientation of \overline{N}^\pm , gives the orientation on M_{cut}^\pm .*

Proof. Fix $x \in Z$. Choose an oriented orthonormal basis for $T_x M$ of the form

$$v_1, \dots, v_{2m-2}, v_\theta, v_N$$

where $v_\theta = c \cdot \left(\frac{\partial}{\partial \theta}\right)_{M,x}$ is a positive multiple of the generating vector field at $x \in Z$ ($c > 0$ is chosen such that v_θ has length 1), $\{v_1, \dots, v_{2m-2}, v_\theta\}$ are an oriented orthonormal basis for $T_x Z$, and v_N is a positively oriented normal vector to Z .

By the definition of the metric and orientation on the reduced space, the push-forward of v_1, \dots, v_{2m-2} by the quotient map $Z \rightarrow Z/S^1$ is an oriented orthonormal basis for $T_{[x]}(Z/S^1)$.

Now the vectors

$$v_1, \dots, v_{2m-2}, 1, i, v_\theta, v_N \in T_{(m,0)} M \times \mathbb{C}$$

are an oriented orthonormal basis, where $1, i \in \mathbb{C}$. Note that $(\frac{\partial}{\partial \theta})_M = (\frac{\partial}{\partial \theta})_{M \times \mathbb{C}}$ on $Z \cong Z \times \{0\} \subset M \times \mathbb{C}$, and that the normal to Z in M can be identified with the normal to \tilde{Z}^\pm in $M \times \mathbb{C}$, when restricted to $Z \subset \tilde{Z}^\pm$. Hence, the push forward of $v_1, \dots, v_{2m-2}, 1, i$ by the quotient map $\tilde{Z}^\pm \rightarrow M_{cut}^\pm$ is an orthonormal basis for $T_{[m,0]}M_{cut}^\pm$.

Since $1, i$ descend to an oriented orthonormal basis for $(\overline{N}^\pm)_x$, when identified with \mathbb{C} using Corollary 7.1.1, the claim follows. \square

7.2 Second lemma - The determinant line bundle.

We would like to relate the determinant line bundles of P_{cut}^\pm (over M_{cut}^\pm), which will be denoted by \mathbb{L}_{cut}^\pm , to the determinant line bundle \mathbb{L} of the spin^c -structure P on M . Denote $\mathbb{L}_{red} = (\mathbb{L}|_Z)/S^1$. This is a line bundle over $Z/S^1 \subset M_{cut}^\pm$.

Then we have:

Lemma 7.2.1. *The restriction of \mathbb{L}_{cut}^\pm to Z/S^1 is isomorphic, as an S^1 -equivariant complex line bundle, to $\mathbb{L}_{red} \otimes \overline{N}^-$.*

Remark 7.2.1. This is not a typo. The restrictions of both \mathbb{L}_{cut}^+ and \mathbb{L}_{cut}^- are isomorphic to $\mathbb{L}_{red} \otimes \overline{N}^-$.

Proof. Recall that the determinant line bundle over the cut spaces is given by

$$\mathbb{L}_{cut}^\pm = [(\mathbb{L} \boxtimes \mathbb{L}_{\mathbb{C}}^\pm)|_{\tilde{Z}^\pm}] / S^1$$

where $\mathbb{L}_{\mathbb{C}}^\pm$ is the determinant line bundle of the spin^c -structure on \mathbb{C} , defined in the process of constructing P_{cut}^\pm , and we divide by the diagonal action of S^1 on $\mathbb{L} \times \mathbb{L}_{\mathbb{C}}^\pm$.

Therefore we have

$$\mathbb{L}_{cut}^\pm|_{Z/S^1} = [(\mathbb{L} \boxtimes \mathbb{L}_{\mathbb{C}}^\pm)|_Z] / S^1 = [\mathbb{L}|_Z \boxtimes \mathbb{L}_{\mathbb{C}}^\pm|_{\{0\}}] / S^1$$

Since the S^1 action on the vector space $\mathbb{L}_{\mathbb{C}}^\pm|_{\{0\}}$ has weight $+1$ (see Remark 4.4.3) we end up with

$$\mathbb{L}_{cut}^\pm|_{Z/S^1} = \mathbb{L}_{red} \otimes (N^-/S^1) = \mathbb{L}_{red} \otimes \overline{N}^-$$

as desired. □

Corollary 7.2.1. *If $F \subset Z/S^1 \subset M_{cut}^\pm$ is a connected component, then S^1 acts on the fibers of $(\mathbb{L}_{cut}^\pm)|_F$ with weight +1.*

Proof. The previous lemma implies that

$$\mathbb{L}_{cut}^\pm|_F = \mathbb{L}_{red}|_F \otimes \overline{N}^-|_F.$$

The action of S^1 on \mathbb{L}_{red} is trivial. Using the isomorphism $\overline{N}^- \simeq Z \times_{S^1} \mathbb{C}$ from Corollary 7.1.1, we see that the action of S^1 on the fibers of $\overline{N}^-|_F$ will have weight +1. □

7.3 Third lemma - The spaces M_\pm

Recall that $M \setminus Z = M_+ \amalg M_-$ (disjoint union), where $M_\pm \subset M$ are open submanifolds.

We have embedding

$$i_\pm: M_\pm \rightarrow M_{cut}^\pm \quad m \mapsto [m, \sqrt{\pm\phi(m)}]$$

which are equivariant and preserve the orientation (see Proposition 6.1 in [6]). Also recall that, as sets, we have $M_{cut}^\pm = Z/S^1 \amalg M_\pm$.

It is important to note that the embeddings $M_\pm \rightarrow M_{cut}^\pm$ do not preserve the metric. This, however, will not effect our calculations.

Lemma 7.3.1. *The restriction of \mathbb{L} to M_\pm is isomorphic to the restriction of L_{cut}^\pm to M_\pm . In other words,*

$$\mathbb{L}|_{M_\pm} \simeq (\mathbb{L}_{cut}^\pm)|_{M_\pm}$$

Proof. Let

$$\widetilde{M}_\pm = \left\{ (m, \sqrt{\pm\phi(m)}) : m \in M_\pm \right\} \subset \widetilde{Z}^\pm,$$

and let

$$pr_1: M \times \mathbb{C} \rightarrow M \quad , \quad pr_2: M \times \mathbb{C} \rightarrow \mathbb{C}$$

be the projections. Then

$$\mathbb{L}_{cut}^\pm = [(pr_1^*(\mathbb{L}) \otimes pr_2^*(\mathbb{L}_{\mathbb{C}}))|_{\tilde{Z}^\pm}] / S^1$$

and when restricting to M_\pm , we get

$$\mathbb{L}_{cut}^\pm|_{M_\pm} = pr_1^*(\mathbb{L})|_{\tilde{M}_\pm} \otimes pr_2^*(\mathbb{L}_{\mathbb{C}})|_{\tilde{M}_\pm} = \mathbb{L}|_{M_\pm} \otimes pr_2^*(\mathbb{L}_{\mathbb{C}})|_{\tilde{M}_\pm}$$

Since $M_\pm \simeq \tilde{M}_\pm$. The term $pr_2^*(\mathbb{L}_{\mathbb{C}})|_{\tilde{M}_\pm}$ is a trivial equivariant complex line bundle, so we conclude that

$$\mathbb{L}_{cut}^\pm|_{M_\pm} = \mathbb{L}|_{M_\pm} \otimes \mathbb{C} = \mathbb{L}|_{M_\pm}$$

as needed. □

7.4 The proof of additivity under cutting

Using all the preliminary lemmas, we can now prove our main theorem.

Proof of Theorem 7.0.1.

Write $M \setminus Z = M_+ \sqcup M_-$. Because the action $S^1 \curvearrowright Z$ is free, the submanifold $Z \subset M$ is a reducible splitting hypersurface (see §4.4). Every connected component $F \subset M^{S^1}$ of the fixed point set must be a subset of either M_+ or M_- .

Also recall that $M_{cut}^\pm = M_\pm \sqcup Z/S^1$, and the action of S^1 on Z/S^1 is trivial (and hence Z/S^1 is a subset of the fixed point set under the action $S \curvearrowright M_{cut}^\pm$).

Using the Kostant formula (Theorem 6.2.1) we get, for any weight $\beta \in Lie(S^1)^*$,

$$\begin{aligned} \#(\beta, Q(M)) &= \sum_{F \subset M^{S^1}} (-1)^F \cdot \bar{N}_F \left(\beta - \frac{1}{2} \mu_F \right) \\ &= \sum_{F \subset (M_+)^{S^1}} (-1)^F \cdot \bar{N}_F \left(\beta - \frac{1}{2} \mu_F \right) + \sum_{F \subset (M_-)^{S^1}} (-1)^F \cdot \bar{N}_F \left(\beta - \frac{1}{2} \mu_F \right) \end{aligned}$$

where the sum is taken over the connected components of the fixed point sets. For the cut spaces we have the following equalities.

$$\begin{aligned} \#(\beta, Q(M_{cut}^\pm)) &= \sum_{F \subset (M_{cut}^\pm)^{S^1}} (-1)^F \cdot \bar{N}_F \left(\beta - \frac{1}{2} \mu_F \right) \\ &= \sum_{F \subset (M_\pm)^{S^1}} (-1)^F \cdot \bar{N}_F \left(\beta - \frac{1}{2} \mu_F \right) + \sum_{F \subset Z/S^1} (-1)^F \cdot \bar{N}_F \left(\beta - \frac{1}{2} \mu_F \right). \end{aligned}$$

In order to prove additivity, we need to show that

$$\sum_{F \subset Z/S^1 \subset M_{cut}^+} (-1)^F \cdot \bar{N}_F \left(\beta - \frac{1}{2} \mu_F \right) + \sum_{F \subset Z/S^1 \subset M_{cut}^-} (-1)^F \cdot \bar{N}_F \left(\beta - \frac{1}{2} \mu_F \right) = 0.$$

Note that the summands in the two sums above are different. In the first, we regard F as a subset of M_{cut}^+ , and in the second, as a subset of M_{cut}^- .

Choose a connected component $F \subset Z/S^1$. Note that F is oriented by the reduced orientation. Since F can be regarded as a subset of both M_{cut}^+ and M_{cut}^- , we will add a superscript F^\pm to emphasize that F is being thought of as a subspace of the corresponding cut space.

It suffices to show that

$$(*) \quad (-1)^{F^+} \cdot \bar{N}_{F^+} \left(\beta - \frac{1}{2} \mu_{F^+} \right) + (-1)^{F^-} \cdot \bar{N}_{F^-} \left(\beta - \frac{1}{2} \mu_{F^-} \right) = 0$$

Recall that $Z \subset M$ is of (real) codimension 1, and so $Z/S^1 \subset M_{cut}^\pm$ is of (real) codimension 2. Therefore, the normal bundle NF^\pm to Z/S^1 in the cut spaces has rank 2. We can turn the bundles NF^\pm to complex line bundles using Corollary 7.1.1, and then the weight of the action $S^1 \curvearrowright NF^\pm$ will be -1 for NF^+ and $+1$ for NF^- .

This is, however, not good, since in order to write down Kostant's formula, we need our weights to be polarized. Therefore, we will use for NF^- the complex structure coming from the isomorphism

$$NF^- \xrightarrow{\cong} Z \times_{S^1} \mathbb{C} ,$$

and for NF^+ , we will use the complex structure which is *opposite* to the one induced by the isomorphism

$$NF^+ \xrightarrow{\cong} Z \times_{S^1} \mathbb{C} .$$

With this convention, the bundles NF^\pm become isomorphic as equivariant complex line bundles, and the weight of the S^1 -action on those bundles is $+1$.

Also, Lemma 7.2.1 implies that the determinant line bundles \mathbb{L}_{cut}^\pm , when restricted to F , are isomorphic as equivariant complex line bundles, and the weight of the S^1 -action on the fibers of $\mathbb{L}_{cut}^\pm|_F$ is $+1$.

Recall now (see §5.3) that the explicit expression for $\overline{N}_{F^\pm} \left(\beta - \frac{1}{2} \mu_{F^\pm} \right)$ involves only the following ingredients:

- μ_{F^\pm} , which are equal to each other ($\mu_{F^\pm} = +1$), since $\mathbb{L}_{cut}^+|_F \simeq \mathbb{L}_{cut}^-|_F$.
- $c_1(NF^\pm)$, which are equal since NF^\pm are isomorphic as complex line bundle, by our previous remark.
- $\hat{A}(TF)$, which are equal, since $F^+ = F^-$ as manifolds.

This means that the terms \overline{N}_{F^\pm} in equation (*) above are the same.

So all is left is to explain why

$$(-1)^{F^+} + (-1)^{F^-} = 0 .$$

But this follows easily from Claim 7.1.1. This claim implies that the orientation on F^- , followed by the one of NF^- , gives the orientation of M_{cut}^- . Hence, $(-1)^{F^-} = 1$. Since

we switched the original orientation for NF^+ , composing the orientation of F^+ with the one of NF^+ will give the opposite orientation on M_{cut}^+ , and hence $(-1)^{F^+} = -1$. The additivity result follows.

□

Chapter 8

An Example: The Two-Sphere

In this chapter we give an example, which illustrates the additivity of spin^c quantization under cutting.

In this example, the manifold is the standard two-sphere $M = S^2 \subset \mathbb{R}^3$, with the outward orientation and the standard Riemannian structure. The circle group $S^1 \subset \mathbb{C}$ acts effectively on the two sphere by rotations about the z-axis.

We will need the following lemma.

Lemma 8.0.1. *Let M be an oriented Riemannian manifold, on which a Lie group G acts transitively by orientation preserving isometries. Choose a point $x \in M$ and denote by G_x the stabilizer at x and by $\sigma: G_x \rightarrow SO(T_x M)$ the isotropy representation. Then:*

1. G_x acts on $SO(T_x M)$ by $g \cdot A = \sigma(g) \circ A$.

2. The map

$$G \rightarrow M \quad , \quad g \mapsto g \cdot x$$

is a principal G_x -bundle (where G_x acts on G by right multiplication).

3. The principal $SO(T_x M)$ -bundle $G \times_{G_x} SO(T_x M)$ is isomorphic to $SOF(M)$, the bundle of oriented orthonormal frames on M .

Proof. (1) is easy. (2) follows from Proposition B.18 in [2] (with $H = G_x$), together with the fact that G/G_x is diffeomorphic to M . To show (3), consider the map taking an element $[g, A] \in G \times_{G_x} SO(T_x M)$ to the frame $g_* \circ A: T_x M \xrightarrow{\cong} T_{g \cdot x} M$. This map can be easily checked to be an isomorphism of principal $SO(T_x M)$ -bundles. \square

8.1 The trivial S^1 -equivariant spin^c structure on S^2

To define an S^1 -equivariant spin^c -structure on S^2 , one needs to describe the space P and the maps in a commutative diagram of the following form (see Remark 3.2.3).

$$\begin{array}{ccccc}
 S^1 \times P & \longrightarrow & P & \longleftarrow & P \times \text{Spin}^c(2) \\
 \downarrow & & \Lambda \downarrow & & \downarrow \\
 S^1 \times \text{SOF}(S^2) & \longrightarrow & \text{SOF}(S^2) & \longleftarrow & \text{SOF}(S^2) \times \text{SO}(2) \\
 \downarrow & & \pi \downarrow & & \\
 S^1 \times S^2 & \longrightarrow & S^2 & &
 \end{array}$$

Set $P = \text{Spin}^c(3)$. By the above lemma, the choice of a point $x = (0, 0, 1) \in S^2$ and a basis for $T_x S^2$ give an isomorphism between the frame bundle of S^2 and $\text{SO}(3) \times_{\text{SO}(2)} \text{SO}(2) = \text{SO}(3)$. Thus $\text{SOF}(S^2) \cong \text{SO}(3)$, and our diagram becomes

$$\begin{array}{ccccc}
 S^1 \times \text{Spin}^c(3) & \longrightarrow & \text{Spin}^c(3) & \longleftarrow & \text{Spin}^c(3) \times \text{Spin}^c(2) \\
 \downarrow & & \Lambda \downarrow & & \downarrow \\
 S^1 \times \text{SO}(3) & \longrightarrow & \text{SO}(3) & \longleftarrow & \text{SO}(3) \times \text{SO}(2) \\
 \downarrow & & \pi \downarrow & & \\
 S^1 \times S^2 & \longrightarrow & S^2 & &
 \end{array}$$

Now we describe the maps in this diagram. Denote

$$C_\theta = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The map $S^1 \times S^2 \rightarrow S^2$ is rotation about the vertical axis, i.e., $(e^{i\theta}, v) \mapsto C_\theta \cdot v$.

The second horizontal row gives the actions of S^1 and $SO(2)$ on the frame bundle $SO(3)$. Those are given by left and right multiplication by C_θ , respectively. The covering map $\pi: SO(3) \rightarrow S^2$ is given by $A \mapsto A \cdot x$, and Λ is the natural map from the spin^c group to the special orthogonal group.

All is left is to describe the actions of S^1 and $\text{Spin}^c(2)$ on $\text{Spin}^c(3)$ (the top row in the diagram). Since $\text{Spin}^c(2) \subset \text{Spin}^c(3)$, this group will act by right-multiplication. The S^1 -action on $\text{Spin}^c(3)$ is given by

$$(e^{i\theta}, [A, z]) \mapsto [x_{\theta/2} \cdot A, e^{i\theta/2} \cdot z] \quad (8.1)$$

where $x_\theta = \cos \theta + \sin \theta \cdot e_1 e_2 \in \text{Spin}(3)$. Note that $x_{\theta/2}$ and $e^{i\theta/2}$ are defined only up to sign, but the equivalence class $[x_{\theta/2}, e^{i\theta/2}]$ is a well defined element in $\text{Spin}^c(3)$.

We will call this S^1 -equivariant spin^c structure *the trivial spin^c -structure on the S^1 -manifold S^2* , and denote it by P_0 . The reason for using the word ‘trivial’ is justified by the following lemma.

Lemma 8.1.1. *The determinant line bundle of the trivial spin^c -structure P_0 is isomorphic to the trivial complex line bundle $\mathbb{L} \cong S^2 \times \mathbb{C}$, with the non-trivial S^1 -action*

$$S^1 \times \mathbb{L} \rightarrow \mathbb{L} \quad , \quad (e^{i\theta}, (v, z)) \mapsto (C_\theta \cdot v, e^{i\theta} \cdot z)$$

Proof. It is easy to check that the map

$$\mathbb{L} = \text{Spin}^c(3) \times_{\text{Spin}^c(2)} \mathbb{C} \rightarrow S^2 \times \mathbb{C} \quad , \quad [[A, z], w] \mapsto (\lambda(A) \cdot x, z^2 w) \quad ,$$

where $\lambda: \text{Spin}(3) \rightarrow SO(3)$ is the double cover and $x = (0, 0, 1)$ is the north pole, is an isomorphism of complex line bundles. The fact that S^1 acts on \mathbb{L} via (8.1), and that

$\lambda(x_{\theta/2}) = C_\theta$, implies that the S^1 action on $S^2 \times \mathbb{C}$, induced by the above isomorphism, is the one stated in the lemma. \square

Another reason for calling P_0 a trivial spin^c -structure, is that the quantization $Q(S^2)$ (with respect to P_0) is the zero space. We do not prove this fact now, since it will follow from a more general statement (see Claim 8.2.3).

8.2 Classifying all spin^c structures on S^2 .

Quantizing the trivial spin^c -structure on S^2 is not interesting, since the quantization is the zero space. However, once we have an equivariant spin^c -structure on a manifold, we can generate all the other equivariant spin^c structures by twisting it with complex equivariant Hermitian line bundles (or, equivalently, with equivariant principal $U(1)$ -bundles). For details on this process, see Appendix D, §2.7 in [2]. We will use this technique to construct all spin^c structures on our S^1 -manifold S^2 .

It is known that all (non-equivariant) complex Hermitian line bundles over S^2 are classified by $H^2(S^2; \mathbb{Z}) \cong \mathbb{Z}$, i.e., by the integers. The S^1 -equivariant line bundles over S^2 are classified by a pair of integers (for instance, the weights of the S^1 -action on the fibers at the poles). This is well known, but because we couldn't find a direct reference, we will give a direct proof of this fact.

Here is an explicit construction of an equivariant line bundle over S^2 , determined by a pair of integer.

Definition 8.2.1. Given a pair of integers (k, n) , define an S^1 -equivariant complex Hermitian line bundle $L_{k,n}$ as follows:

1. As a complex line bundle,

$$L_{k,n} = \text{Spin}(3) \times_{\text{Spin}(2)} \mathbb{C} \cong S^3 \times_{S^1} \mathbb{C} ,$$

where $\text{Spin}(2) \cong S^1$ acts on \mathbb{C} with weight n and on $\text{Spin}(3)$ by right multiplication.

2. The circle group S^1 acts on $L_{k,n}$ by

$$S^1 \times L_{k,n} \rightarrow L_{k,n} \quad , \quad (e^{i\theta}, [A, z]) \mapsto [x_{\theta/2} \cdot A, e^{\frac{i\theta}{2}(n+2k)} \cdot z]$$

where $x_\theta = \cos \theta + \sin \theta \cdot e_1 e_2 \in Spin(2) \subset Spin(3)$.

And now we prove:

Claim 8.2.1. *Every S^1 equivariant line bundle over S^2 is isomorphic to $L_{k,n}$, for some $k, n \in \mathbb{Z}$.*

Proof. Let L be an S^1 -equivariant line bundle over S^2 . Since L is, in particular, an ordinary line bundle, we can assume it is of the form $L = S^3 \times_{S^1} \mathbb{C}$ where S^1 acts on \mathbb{C} with weight n . Also, since L is an equivariant line bundle, we have a map

$$\rho: S^1 \times L \rightarrow L \quad , \quad (e^{i\theta}, x) \mapsto e^{i\theta} \cdot x .$$

Define a map

$$\eta: S^1 \times L \rightarrow L \quad , \quad (e^{i\theta}, [A, z]) \mapsto [x_{-\theta/2} \cdot A, e^{-i\theta n/2} z] .$$

This map is well defined.

By composing ρ and η we get a third map

$$\delta: S^1 \times L \rightarrow L$$

which lifts the trivial action on S^2 . Since S^2 is connected, this composed action will act on all the fibers of L with one fixed weight k . Therefore, we get

$$e^{i\theta} \cdot [x_{-\theta/2} \cdot A, e^{-i\theta n/2} z] = [A, e^{ik\theta} z]$$

and after a change of variables, the given action $S^1 \curvearrowright L$ is

$$e^{i\theta} \cdot [B, w] = [x_{\theta/2} \cdot B, e^{i\theta n/2 + ik\theta} w] .$$

This means that L is isomorphic to $L_{k,n}$. □

We now ‘twist’ the trivial spin^c -structure by $U(L_{k,n})$, the unit circle bundle of $L_{k,n}$, to get nontrivial spin^c structures on S^2 . Observe that the group $U(1)$ acts on $\text{Spin}^c(3)$ from the right by multiplication by elements of the form $[1, c] \in \text{Spin}^c(3)$.

Definition 8.2.2.

$$P_{k,n} = P_0 \times_{U(1)} U(L_{k,n})$$

where we quotient by the anti-diagonal action of $U(1)$.

This is an S^1 -equivariant spin^c -structure on S^2 . The principal action of $\text{Spin}^c(2)$ comes from acting from the right on the $P_0 \cong \text{Spin}^c(3)$ component, and the left S^1 -action is induced from the diagonal action on $P_0 \times L_{k,n}$.

Claim 8.2.2. Fix $(k, n) \in \mathbb{Z}^2$, and denote by $\mathbb{L} = \mathbb{L}_{k,n}$ the determinant line bundle associated to the spin^c -structure $P_{k,n}$ on S^2 . Let $N = (0, 0, 1)$, $S = (0, 0, -1) \in S^2$ be the north and the south poles.

Then S^1 acts on $\mathbb{L}|_N$ with weight $2k + 2n + 1$ and on $\mathbb{L}|_S$ with weight $2k + 1$.

Proof. The determinant line bundle is

$$\mathbb{L} = P_{k,n} \times_{\text{Spin}^c(2)} \mathbb{C} = [\text{Spin}^c(3) \times_{U(1)} (S^3 \times_{S^1} S^1)] \times_{\text{Spin}^c(2)} \mathbb{C}.$$

An element of \mathbb{L} can be written in the form $[[[A, 1], [A, 1]], u]$, where $A \in \text{Spin}(3) \cong S^3$ and $u \in \mathbb{C}$.

1. For the north pole $N = (0, 0, 1)$, can choose $A = 1 \in \text{Spin}(3)$, hence an element of $\mathbb{L}|_N$ will have the form $[[[1, 1], [1, 1]], u]$. Let $e^{i\theta} \in S^1$, act on $\mathbb{L}|_N$, to get

$$\begin{aligned} & [[[x_{\theta/2}, e^{i\theta/2}], [x_{\theta/2}, e^{i\theta(n+2k)/2}]], u] = [[[1, 1], [x_{\theta/2}, e^{i\theta(n+2k)/2}]], e^{i\theta} u] = \\ & = [[[1, 1], [x_{\theta/2}, e^{i\theta \cdot n/2}]], e^{i\theta(1+2k)} u] = [[[1, 1], [1, e^{i\theta \cdot n/2} \cdot e^{i\theta \cdot n/2}]], e^{i\theta(1+2k)} u] = \\ & = [[[1, 1], [1, 1]], e^{i\theta(1+2k+2n)} u] \end{aligned}$$

and therefore the weight on $\mathbb{L}|_N$ is $1 + 2k + 2n$.

2. For the south pole $S = (0, 0, -1)$ choose $A = e_2 e_3$. We compute again the action of an element $e^{i\theta}$ on $[[[A, 1], [A, 1]], u]$, and use the identity $A \cdot x_\theta = x_{-\theta} \cdot A$ for any $x_\theta \in Spin(2) \subset Spin(3)$.

$$\begin{aligned} & [[x_{\theta/2} A, e^{i\theta/2}], [x_{\theta/2} A, e^{i\theta(n+2k)/2}], u] = [[A, 1], [Ax_{-\theta/2}, e^{i\theta(n+2k)/2}], e^{i\theta} u] = \\ & = [[A, 1], [Ax_{-\theta/2}, e^{i\theta \cdot n/2}], e^{i\theta(1+2k)} u] = \\ & = [[A, 1], [A, e^{-i\theta \cdot n/2} \cdot e^{i\theta \cdot n/2}], e^{i\theta(1+2k)} u] = [[A, 1], [A, 1], e^{i\theta(1+2k)} u] \end{aligned}$$

and therefore the weight on $\mathbb{L}|_S$ is $2k + 1$.

□

Remark 8.2.1. Note that the $2k + 2n + 1$ and $2k + 1$ are both odd numbers. This is not surprising in view of Lemma 5.0.1. The isotropy weight at N (or at S) is ± 1 and its sum with the weight on \mathbb{L}_N (or on \mathbb{L}_S) must be even. This implies that the weights of $S^1 \circlearrowleft \mathbb{L}_{\{N,S\}}$ must be odd.

Remark 8.2.2. The above claim implies that the determinant line bundle of the $spin^c$ -structure $P_{k,n}$ is isomorphic to $L_{2k+1,2n}$, i.e., $\mathbb{L}_{k,n} \cong L_{2k+1,2n}$.

Claim 8.2.3. Fix $(k, n) \in \mathbb{Z}^2$ and denote by $Q_{k,n}(S^2)$ the quantization of the $spin^c$ structure $P_{k,n}$ on S^2 . Then the multiplicity of a weight $\beta \in Lie(S^1)^* \cong \mathbb{R}$ in $Q_{k,n}(S^2)$ is given by

$$\#(\beta, Q_{k,n}(S^2)) = \begin{cases} 1 & 0 < \beta - k \leq n \\ -1 & n < \beta - k \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

In particular, if $n = 0$, then $Q_{k,0}(S^2)$ is the zero representation.

Proof. By the Kostant formula for $spin^c$ -quantization (Theorem 5.0.1) the multiplicity is

given by

$$\#(\beta, Q_{k,n}(S^2)) = \overline{N}_{(0,0,1)} \left(\beta - \frac{1+2k+2n}{2} \right) - \overline{N}_{(0,0,-1)} \left(\beta - \frac{1+2k}{2} \right) .$$

The definition of \overline{N}_p implies that

$$\overline{N}_{(0,0,1)} \left(\beta - \frac{1+2k+2n}{2} \right) = \begin{cases} 1 & \beta - k \leq n \\ 0 & \beta - k > n \end{cases}$$

and similarly,

$$\overline{N}_{(0,0,-1)} \left(\beta - \frac{1+2k}{2} \right) = \begin{cases} 1 & \beta - k \leq 0 \\ 0 & \beta - k > 0 \end{cases} .$$

Using that, one can compute $\#(\beta, Q_{k,n}(S^2))$ and get the required result. \square

8.3 Cutting a spin^c -structure on S^2

Now we get to the cutting of the spin^c structure $P_{k,n}$ on S^2 . Let \mathbb{L} be the determinant line bundle of $P_{k,n}$. We take the equator $Z = \{(\cos \alpha, \sin \alpha, 0)\} \subset S^2$ to be our reducible splitting hypersurface (see §4.4). The cut spaces M_{cut}^\pm are both diffeomorphic to S^2 , and we would like to know what are $(P_{k,n})_{cut}^\pm$. Because the cut spaces are spheres again, we must have

$$(P_{k,n})_{cut}^\pm = P_{k^\pm, n^\pm} \quad \text{for some integers} \quad k^\pm, n^\pm .$$

Corollary 7.2.1 implies that S^1 acts on $\mathbb{L}_{cut}^-|_N$ and on $\mathbb{L}_{cut}^+|_S$ with weight $+1$. Lemma 7.3.1 implies that the weight of the S^1 action on $\mathbb{L}|_N$ and $\mathbb{L}|_S$ will be equal to the weight of the action on $\mathbb{L}_{cut}^+|_N$ and $\mathbb{L}_{cut}^-|_S$, respectively. From this we get the equations

$$2k^+ + 1 = 1 \quad , \quad 2k^+ + 2n^+ + 1 = 2k + 2n + 1 \quad , \quad 2k^- + 2n^- + 1 = 1 \quad , \quad 2k^- + 1 = 2k + 1$$

which yield $k^+ = 0$, $n^+ = k + n$, $k^- = k$, $n^- = -k$. Therefore we obtain:

$$(P_{k,n})_{cut}^+ = P_{0, k+n} \quad , \quad (P_{k,n})_{cut}^- = P_{k, -k} .$$

Remark 8.3.1. We see that there is no symmetry between the spin^c structures on the ‘+’ and ‘-’ cut spaces as one might expect. This is because the definition of the covering map $SO(3) \rightarrow S^2$ involved a choice of a point (in our case - the north pole), which ‘broke’ the symmetry of the two-sphere.

The quantization of the cut spaces is thus obtained from Claim 8.2.3. For the ‘+’ cut space we get, for any weight $\beta \in \mathbb{Z}$:

$$\#(\beta, Q_{k,n}^+(S^2)) = \#(\beta, Q_{0,k+n}(S^2)) = \begin{cases} 1 & -k < \beta - k \leq n \\ -1 & n < \beta - k \leq -k \\ 0 & \text{otherwise} \end{cases}$$

and for the ‘-’ cut space:

$$\#(\beta, Q_{k,n}^-(S^2)) = \#(\beta, Q_{k,-k}(S^2)) = \begin{cases} 1 & 0 < \beta - k \leq -k \\ -1 & -k < \beta - k \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

It is an easy exercise to check that

$$\#(\beta, Q_{k,n}(S^2)) = \#(\beta, Q_{k,n}^+(S^2)) + \#(\beta, Q_{k,n}^-(S^2))$$

and this implies that as virtual S^1 -representations, we have

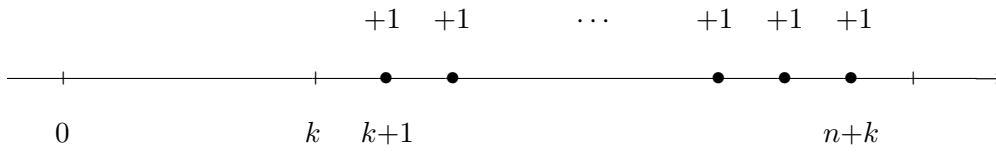
$$Q_{k,n}(S^2) = Q_{k,n}^-(S^2) \oplus Q_{k,n}^+(S^2) .$$

As expected, we have additivity of spin^c quantization under cutting in this example.

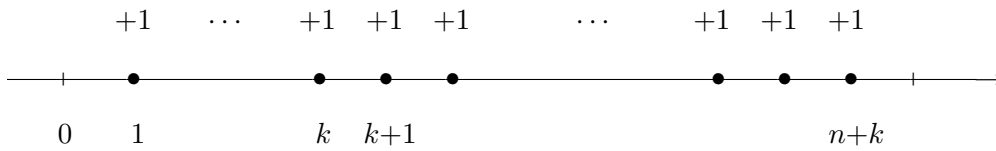
8.4 Multiplicity diagrams

The S^1 -equivariant spin^c quantization of a manifold M can be described using multiplicity diagrams as follows. Above each integer on the real line, we write the multiplicity of the weight represented by this integer, if it is nonzero.

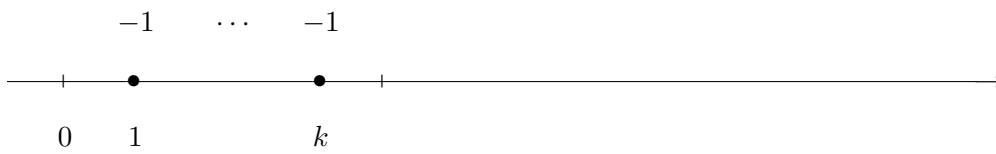
For example, if $n, k > 0$, then the quantization $Q_{k,n}$ of S^2 is given by the following diagram.



The quantization of the '+' cut space, $Q_{k,n}^+$, which is equal to $Q_{0,k+n}$, will have the following diagram.

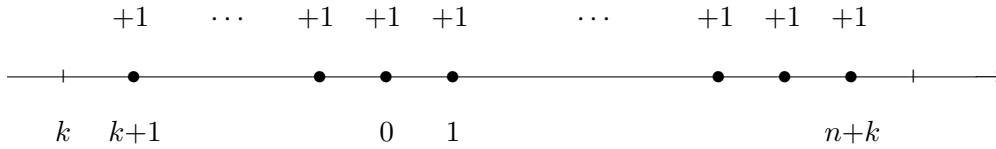


Finally, $Q_{k,n}^- = Q_{k,-k}$ is given by

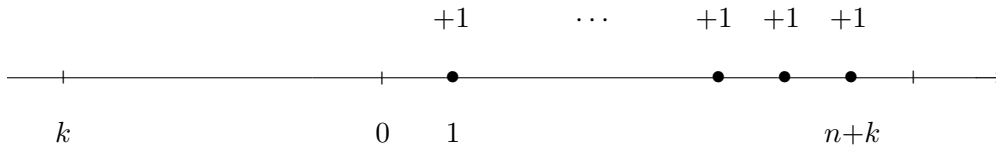


Clearly, one can see that the diagram of $Q_{k,n}$ is the 'sum' of the diagrams of $Q_{k,n}^\pm$.

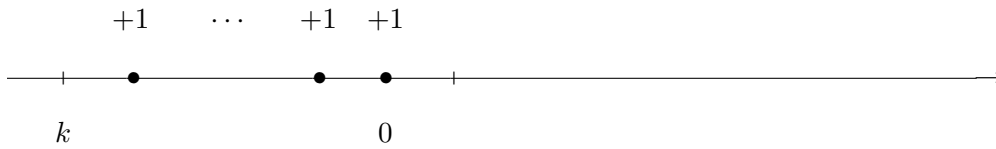
Let us present another case, where only positive multiplicities occur in the quantization of all three spaces (the original manifold S^2 and the cut spaces). This happens if $k < 0 < n + k$. In this case, the diagram for $Q_{k,n}$ is as follows.



The diagram for $Q_{k,n}^+ = Q_{0,n+k}$ is



and for $Q_{k,n}^- = Q_{k,-k}$ we have



and again the additivity is clear.

The additivity is clearer in the last set of diagrams, as we can actually see the diagram of $Q_{k,n}$ being cut into two parts. It seems like the diagram was cut at some point between 0 and 1 . The point at which the cutting is done depends on the spin^c -structure on \mathbb{C} that was chosen during the cutting process (see §4.4).

Part II

Spin^c Prequantization and Symplectic Cutting

Chapter 9

Introduction to Part II

Given a compact even-dimensional oriented Riemannian manifold M , endowed with a spin^c -structure, one can construct an associated Dirac operator D^+ acting on smooth sections of a certain (complex) vector bundle over M . The spin^c quantization of M with respect to the above structure is defined to be

$$Q(M) = \ker(D^+) - \text{coker}(D^+) .$$

This is a virtual vector space, and in the presence of a G -action, it is a virtual representation of the group G . Spin^c quantization generalizes the concept of *Kähler* and *almost-complex quantization* (see [6], especially Lemma 2.7 and Remark 2.9) and in some sense it is a ‘better behaved’ quantization (see Part I of this thesis).

Quantization was originally defined as a process that associates a Hilbert space to a symplectic manifold (and self-adjoint operators to smooth real valued functions on the manifold). Therefore, one of our goals in this part is to relate spin^c quantization to symplectic geometry. This can be achieved by defining a *spin^c -prequantization* of a symplectic manifold to be a spin^c -structure and a connection on its determinant line bundle which are compatible with the symplectic form (in a certain sense). This definition is analogous to the definition of prequantization in the context of geometric quantization (see [13] and references therein). Our definition is different but equivalent to the one in

[6]. It is important to mention that in the equivariant setting, a spin^c -prequantization for a symplectic manifold (M, ω) determines a moment map $\Phi: M \rightarrow \mathfrak{g}^*$, and hence the action $G \curvearrowright (M, \omega)$ is Hamiltonian.

The cutting construction was originally introduced by E. Lerman in [4] for symplectic manifolds equipped with a Hamiltonian circle action. In Chapter 8 we explained how one can cut a given S^1 -equivariant spin^c -structure on an oriented Riemannian manifold. Here we extend this construction and describe how to cut a given S^1 -equivariant spin^c -prequantization. This cutting process involves two choices: a choice of an equivariant spin^c -prequantization for the complex plane \mathbb{C} , and a choice of a level set $\Phi^{-1}(\alpha)$ along which the cutting is done. Our main theorem (Theorem 11.4.1) reveals a quite interesting fact: Those two choices must be compatible (in a certain sense) in order to make the cutting construction possible. In fact, each one of the two choices determines the other (once we assume that cutting is possible), so in fact only one choice is to be made. This theorem also explains the ‘mysterious’ freedom one has when choosing a spin^c -structure on \mathbb{C} in the first step of the cutting construction: it is just the freedom of choosing a ‘cutting point’ $\alpha \in \mathfrak{g}^*$ (or a level set of the moment map along which the cutting is done). Since by our theorem, α can never be a weight, we see why spin^c quantization must be additive under cutting (as proved in Chapter 7).

This part is organized as follows. In Chapter 10 we review the definitions of the spin groups, spin and spin^c structures and define the concept of spin^c -prequantization. As an example we will use later, we construct a prequantization for the complex plane. For technical reasons, we chose to define spin^c prequantization for manifold endowed with closed two-forms (which may not be symplectic). In Chapter 11 we describe the cutting process in steps and obtain our main theorem relating the spin^c -prequantization for \mathbb{C} with the level set used for cutting. In Chapters 12 and 13 we discuss a couple of examples.

Chapter 10

Spin^c Prequantization

10.1 Spin^c structures

In this section we recall the definition and basic properties of the spin and spin^c groups. Then we give the definition of a spin^c structure on a manifold, which is essential for defining spin^c-prequantization.

Definition 10.1.1. Let V be a finite dimensional vector space over $\mathbb{K} = \mathbb{R}$ or \mathbb{C} , equipped with a symmetric bilinear form $B : V \times V \rightarrow \mathbb{K}$. Define the *Clifford algebra* $Cl(V, B)$ to be the quotient $T(V)/I(V, B)$ where $T(V)$ is the tensor algebra of V , and $I(V, B)$ is the ideal generated by $\{v \otimes v - B(v, v) \cdot 1 : v \in V\}$.

Remark 10.1.1. If v_1, \dots, v_n is an orthogonal basis for V , then $Cl(V, B)$ is the algebra generated by v_1, \dots, v_n , subject to the relations $v_i^2 = B(v_i, v_i) \cdot 1$ and $v_i v_j = -v_j v_i$ for $i \neq j$.

Also note that V is a vector subspace of $Cl(V, B)$.

Definition 10.1.2. If $V = \mathbb{R}^k$ and B is minus the standard inner product on V , then define the following objects:

1. $C_k = Cl(V, B)$, and $C_k^c = Cl(V, B) \otimes \mathbb{C}$.

Those are finite dimensional algebras over \mathbb{R} and \mathbb{C} , respectively.

2. The *spin* group

$$Spin(k) = \{v_1 v_2 \dots v_l : v_i \in \mathbb{R}^k, \|v_i\| = 1 \text{ and } 0 \leq l \text{ is even}\} \subset C_k$$

3. The *spin^c* group

$$Spin^c(k) = (Spin(k) \times U(1)) / K$$

where $U(1) \subset \mathbb{C}$ is the unit circle, and $K = \{(1, 1), (-1, -1)\}$.

Remark 10.1.2.

1. Equivalently, one can define

$$\begin{aligned} Spin^c(k) &= \\ &= \{c \cdot v_1 \cdots v_l : v_i \in \mathbb{R}^k, \|v_i\| = 1, 0 \leq l \text{ is even, and } c \in U(1)\} \subset C_k^c \end{aligned}$$

2. The group $Spin(k)$ is connected for $k \geq 2$.

Proposition 10.1.1.

1. There is a linear map $C_k \rightarrow C_k$, $x \mapsto x^t$ characterized by $(v_1 \dots v_l)^t = v_l \dots v_1$ for all $v_1, \dots, v_l \in \mathbb{R}^k$.
2. For each $x \in Spin(k)$ and $y \in \mathbb{R}^k$, we have $xyx^t \in \mathbb{R}^k$.
3. For each $x \in Spin(k)$, the map $\lambda(x) : \mathbb{R}^k \rightarrow \mathbb{R}^k$, $y \mapsto xyx^t$ is in $SO(k)$, and $\lambda : Spin(k) \rightarrow SO(k)$ is a double covering for $k \geq 1$. It is a universal covering map for $k \geq 3$.

For the proof, see page 16 in [1].

Definition 10.1.3. Let M be a manifold, and Q a principal $SO(k)$ -bundle on M . A *spin^c structure* on Q is a principal $Spin^c(k)$ -bundle $P \rightarrow M$, together with a map $\Lambda : P \rightarrow Q$ such that the following diagram commutes.

$$\begin{array}{ccc}
P \times \text{Spin}^c(k) & \longrightarrow & P \\
\downarrow \Lambda \times \lambda^c & & \downarrow \Lambda \\
Q \times \text{SO}(k) & \longrightarrow & Q
\end{array}$$

Here, the maps corresponding to the horizontal arrows are the principal actions, and $\lambda^c : \text{Spin}^c(k) \rightarrow \text{SO}(k)$ is given by $[x, z] \mapsto \lambda(x)$, where $\lambda : \text{Spin}(k) \rightarrow \text{SO}(k)$ is the double covering.

Remark 10.1.3.

1. A spin^c -structure on an oriented Riemannian vector bundle E is a spin^c -structure on the associated bundle of oriented orthonormal frames, $\text{SOF}(E)$.
2. A spin^c -structure on an oriented Riemannian manifold is a spin^c -structure on its tangent bundle.

10.2 Equivariant spin^c structures

Definition 10.2.1. Let G, H be Lie groups. A G -equivariant principal H -bundle is a principal H -bundle $\pi : Q \rightarrow M$ together with left G -actions on Q and M , such that:

1. $\pi(g \cdot q) = g \cdot \pi(q)$ for all $g \in G$, $q \in Q$
(i.e., G acts on the fiber bundle $\pi : Q \rightarrow M$).
2. $(g \cdot q) \cdot h = g \cdot (q \cdot h)$ for all $g \in G$, $q \in Q$, $h \in H$
(i.e., the actions of G and H commute).

Remark 10.2.1. It is convenient to think of a G -equivariant principal H -bundle in terms of the following commuting diagram (the horizontal arrows correspond to the G and H actions).

$$\begin{array}{ccccc}
 G \times Q & \longrightarrow & Q & \longleftarrow & Q \times H \\
 \text{Id} \times \pi \downarrow & & \downarrow \pi & & \\
 G \times M & \longrightarrow & M & &
 \end{array}$$

Definition 10.2.2. Let $\pi : E \rightarrow M$ be a fiberwise oriented Riemannian vector bundle, and let G be a Lie group. A G -equivariant structure on E is an action of G on the vector bundle, that preserves the orientations and the inner products of the fibers. We will say that E is a G -equivariant oriented Riemannian vector bundle.

Remark 10.2.2.

1. A G -equivariant oriented Riemannian vector bundle E over a manifold M , naturally turns $\text{SOF}(E)$ into a G -equivariant principal $\text{SO}(k)$ -bundle, where $k = \text{rank}(E)$.
2. If a Lie group G acts on an oriented Riemannian manifold M , by orientation preserving isometries, then the frame bundle $\text{SOF}(M)$ becomes a G -equivariant principal $\text{SO}(m)$ -bundle, where $m = \dim(M)$.

Definition 10.2.3. Let $\pi : Q \rightarrow M$ be a G -equivariant principal $\text{SO}(k)$ -bundle. A G -equivariant spin^c -structure on Q is a spin^c structure $\Lambda : P \rightarrow Q$ on Q , together with a left action of G on P , such that

1. $\Lambda(g \cdot p) = g \cdot \Lambda(p)$ for all $p \in P, g \in G$ (i.e., G acts on the bundle $P \rightarrow Q$).
2. $g \cdot (p \cdot x) = (g \cdot p) \cdot x$ for all $g \in G, p \in P, x \in \text{Spin}^c(k)$
(i.e., the actions of G and $\text{Spin}^c(k)$ on P commute).

Remark 10.2.3.

1. It is convenient to think of a G -equivariant spin^c structure in terms of the following commuting diagram (where the horizontal arrows correspond to the principal and the G -actions).

$$\begin{array}{ccccc}
 G \times P & \longrightarrow & P & \longleftarrow & P \times \text{Spin}^c(k) \\
 \text{Id} \times \Lambda \downarrow & & \Lambda \downarrow & & \Lambda \times \lambda^c \downarrow \\
 G \times Q & \longrightarrow & Q & \longleftarrow & Q \times \text{SO}(k) \\
 \text{Id} \times \pi \downarrow & & \pi \downarrow & & \\
 G \times M & \longrightarrow & M & &
 \end{array}$$

2. Note that in a G -equivariant spin^c structure, the bundle $P \rightarrow M$ is a G -equivariant principal $\text{Spin}^c(k)$ -bundle.

10.3 The definition of spin^c prequantization

In this section we define the concept of a G -equivariant Spin^c prequantization. This will consist of a G -equivariant spin^c structure and a connection on the corresponding $U(1)$ -bundle, which is compatible with a given two-form on our manifold. To motivate the definition, we begin by proving the following claim.

Claim 10.3.1. *Let M be a compact oriented Riemannian manifold of dimension $2m$, on which a Lie group G acts by orientation preserving isometries, and let $P \rightarrow \text{SO}(M) \rightarrow M$ be a G -equivariant spin^c structure on M .*

Assume that $\theta: TP \rightarrow \mathfrak{u}(1) \cong i\mathbb{R}$ is a G -invariant and $\text{Spin}^c(m)$ -invariant connection 1-form on the principal S^1 -bundle $\pi: P \rightarrow \text{SO}(M)$, for which

$$\theta(\zeta_P): P \rightarrow \mathfrak{u}(1)$$

is a constant function for any $\zeta \in \mathfrak{spin}(m)$.

For each $\xi \in \mathfrak{g} = \text{Lie}(G)$ define a map

$$\phi^\xi: P \rightarrow \mathbb{R} \quad , \quad \phi^\xi = -i \cdot (\iota_{\xi_P} \theta)$$

where ξ_P is the vector field on P generated by ξ .

Then

1. For any $\xi \in \mathfrak{g}$, the map ϕ^ξ is $\text{Spin}^c(2m)$ -invariant, i.e., $\phi^\xi = \pi^*(\Phi^\xi)$ where $\Phi^\xi: M \rightarrow \mathbb{R}$ is a smooth function.
2. For any $\xi \in \mathfrak{g}$, we have $d\Phi^\xi = \iota_{\xi_M}\omega$, where ω is a (necessarily closed) real two-form on M , determined by the equation $d\theta = \pi^*(-i \cdot \omega)$.
3. The map

$$\Phi: M \rightarrow \mathfrak{g}^* \quad , \quad \Phi(m)\xi = \Phi^\xi(m)$$

is G -equivariant.

Proof.

1. This follows from the fact that θ is $\text{Spin}^c(m)$ -invariant, and that the $\text{Spin}^c(m)$ and G -actions on P commute.
2. For any $\eta = (\zeta, b) \in \mathfrak{spin}^c(m) = \mathfrak{spin}(n) \oplus \mathfrak{u}(1)$, we have

$$\iota_{\eta_P}\theta = \theta(\eta_P) = \theta(\zeta_P) + \theta(b_P) = \theta(\zeta_P) + b .$$

Since $\theta(\zeta_P)$ is constant by assumption, we get that

$$\iota_{\eta_P}d\theta = L_{\eta_P}\theta - d\iota_{\eta_P}\theta = 0 .$$

This implies that $d\theta$ is horizontal, and hence ω is well defined by the equation $d\theta = \pi^*(-i \cdot \omega)$. It is closed since π^* is injective.

Now, observe that

$$\begin{aligned} \pi^*d\Phi^\xi &= d(\pi^*\Phi^\xi) = d\phi^\xi = -i d\iota_{\xi_P}\theta = -i [L_{\xi_P}\theta - \iota_{\xi_P}d\theta] = \\ &= \iota_{\xi_P}(\pi^*\omega) = \pi^*(\iota_{\xi_M}\omega) \end{aligned}$$

and since π^* is injective, we get $d\Phi^\xi = \iota_{\xi_M}\omega$ as needed.

3. If $g \in G$, $m \in M$, $\xi \in \mathfrak{g}$ and $p \in \pi^{-1}(m)$, then

$$\begin{aligned} \Phi^\xi(g \cdot m) &= \phi^\xi(g \cdot p) = -i(\iota_{\xi_P} \theta)(g \cdot p) = -i(\theta_{g \cdot p}(\xi_P|_{g \cdot p})) = \\ &= -i(\theta_{g \cdot p}(g \cdot (Ad_{g^{-1}} \xi)_P|_p)) = -i\left(\iota_{(Ad_{g^{-1}} \xi)_P} \theta\right)(p) = \\ &= \phi^{Ad_{g^{-1}} \xi}(p) = \Phi^{Ad_{g^{-1}} \xi}(m) \end{aligned}$$

and we ended up with $\Phi^\xi(g \cdot m) = \Phi^{Ad_{g^{-1}} \xi}(m)$, which means that Φ is G -equivariant. □

The above claim suggests a compatibility condition between a given two-form and a spin^c structure on our manifold. We will work with two-forms that are closed, but not necessarily nondegenerate. The compatibility condition is formulated in the following definition.

Definition 10.3.1. Let a Lie group G act on a m -dimensional manifold M , and let ω be a G -invariant closed two-form (i.e., $g^* \omega = \omega$ for any $g \in G$). A G -equivariant spin^c prequantization for M is a G -equivariant spin^c -structure $\pi: P \rightarrow \text{SOF}(M) \rightarrow M$ (with respect to an invariant Riemannian metric and orientation), and a G and $\text{Spin}^c(m)$ -invariant connection $\theta \in \Omega^1(P; \mathfrak{u}(1))$ on $P \rightarrow \text{SOF}(M)$, such that

$$\theta(\zeta_P) = 0 \quad \text{for any } \zeta \in \mathfrak{spin}(m)$$

and

$$d\theta = \pi^*(-i \cdot \omega) .$$

Remark 10.3.1. By the above claim, the action $G \curvearrowright (M, \omega)$ is Hamiltonian, with a moment map $\Phi: M \rightarrow \mathfrak{g}^*$ satisfying

$$\pi^*(\Phi^\xi) = -i \cdot \iota_{\xi_P}(\theta) \quad \text{for any } \xi \in \mathfrak{g} .$$

Remark 10.3.2. A G -invariant connection 1-form θ on the G -equivariant principal $\text{Spin}^c(m)$ -bundle $P \rightarrow M$ induces a connection 1-form $\tilde{\theta}$ on the principal S^1 -bundle $P \rightarrow \text{SOF}(M)$ as follows.

Recall the determinant map

$$\det: \text{Spin}^c(n) \rightarrow U(1) \quad , \quad [A, z] \mapsto z^2 .$$

This map induces a map on the Lie algebras

$$\det_*: \mathfrak{spin}^c(n) = \mathfrak{spin}(n) \oplus \mathfrak{u}(1) \rightarrow \mathfrak{u}(1) \simeq i\mathbb{R} \quad , \quad (A, z) \mapsto 2z .$$

This means that the map $\frac{1}{2}\det_*: \mathfrak{spin}^c(m) \rightarrow \mathfrak{u}(1)$ is just the projection onto the $\mathfrak{u}(1)$ component.

The composition $\frac{1}{2}\det_* \circ \theta$ will then be a connection 1-form on $P \rightarrow \text{SOF}(M)$, which is G -invariant, and for which $\tilde{\theta}(\zeta_P) = \frac{1}{2}\det_*(\zeta) = 0$ for any $\zeta \in \mathfrak{spin}(m)$.

Remark 10.3.3. The condition $\theta(\zeta_P) = 0$ could have been omitted, since our main theorem can be proved without it. However, this condition is necessary to obtain a discrete condition on the prequantizable closed two forms. See the example in Chapter 12.

In the following claim, M is an oriented Riemannian m -dimensional manifold on which G acts by orientation preserving isometries.

Claim 10.3.2. *Let $P \rightarrow \text{SOF}(M) \rightarrow M$ be a G -equivariant spin^c -structure on M . Let $P_{\det} = P/\text{Spin}(m)$ and $q: P \rightarrow P_{\det}$ the quotient map. Let $\theta: TP \rightarrow \mathfrak{u}(1)$ be a connection 1-form on the G -equivariant principal $U(1)$ -bundle $P \rightarrow \text{SOF}(M)$.*

Then $\theta = \frac{1}{2}q^(\bar{\theta})$ for some connection one form $\bar{\theta}$ on the G -equivariant principal $U(1)$ bundle $P_{\det} \rightarrow M$ if and only if θ is $\text{Spin}^c(m)$ -invariant and $\theta(\zeta_P) = 0$ for all $\zeta \in \mathfrak{spin}(m)$.*

Here is the relevant diagram.

$$\begin{array}{ccc} P & \xrightarrow{q} & P_{\det} \\ \downarrow & & \downarrow \\ \text{SOF}(M) & \longrightarrow & M \end{array}$$

Note that this is not a pullback diagram. The pullback of P_{\det} under the projection $\text{SOF}(M) \rightarrow M$ is the square of the principal $U(1)$ bundle $P \rightarrow \text{SOF}(M)$.

Proof of Claim 10.3.2. Assume that $\theta = \frac{1}{2}q^*(\bar{\theta})$. Then for any $g \in \text{Spin}^c(m): P \rightarrow P$, write $g = [A, z]$ with $A \in \text{Spin}(m)$ and $z \in U(1)$. Since θ is $U(1)$ -invariant, we have

$$g^*\theta = [A, 1]^*[1, z]^*\theta = [A, 1]^*\theta = \frac{1}{2}[A, 1]^*q^*\bar{\theta} = \frac{1}{2}q^*\bar{\theta} = \theta ,$$

and so θ is $\text{Spin}^c(m)$ -invariant. If $\zeta \in \mathfrak{spin}(m)$ then $q_*(\zeta_P) = 0$, which implies $\theta(\zeta_P) = 0$.

Conversely, assume that θ is $\text{Spin}^c(m)$ -invariant with $\theta(\zeta_P) = 0$ for all $\zeta \in \mathfrak{spin}(m)$.

Define a 1-form $TP_{det} \rightarrow \mathfrak{u}(1)$ by

$$\bar{\theta}(q_*v) = 2\theta(v) \quad \text{for } v \in TP .$$

This will be well defined, since if $q_*v = q_*v'$ for $v \in T_xP$ and $v' \in T_{xg}P_{det}$ where $g \in \text{Spin}(m)$, then $q_*(v - v'g^{-1}) = 0$, which implies that $v - v'g^{-1} = \zeta_P$ for some $\zeta \in \mathfrak{spin}(m)$. The fact that $\theta(\zeta_P) = 0$ will imply that $\theta(v) = \theta(v')$. Smoothness and G -invariance of $\bar{\theta}$ are straight forward.

We also need to check that $\bar{\theta}$ is vertical (i.e., that $\bar{\theta}(\xi_{P_{det}}) = \xi$ for $\xi \in \mathfrak{u}(1)$). Note that $\text{Spin}^c(m)/\text{Spin}(m)$ is isomorphic to $U(1)$ via the isomorphism taking the class of $[A, z] \in \text{Spin}^c(m)$ to $z^2 \in U(1)$. This will imply that $q_*(\xi_P) = 2\xi_{P_{det}}$, from which we can conclude that $\bar{\theta}$ is vertical. \square

10.4 Spin^c prequantizations for \mathbb{C}

For the purpose of cutting, we will need to choose an S^1 -equivariant spin^c -prequantization on the complex plane. The S^1 -action on \mathbb{C} is given by

$$(a, z) \mapsto a^{-1} \cdot z \quad , \quad a \in S^1, z \in \mathbb{C} .$$

We take the standard orientation and Riemannian structure on \mathbb{C} and choose our two-form to be

$$\omega_{\mathbb{C}} = 2 \cdot dx \wedge dy = -i \cdot dz \wedge d\bar{z} .$$

For each odd integer $\ell \in \mathbb{Z}$ we will define an S^1 -equivariant spin^c prequantization for $S^1 \curvearrowright (\mathbb{C}, \omega_{\mathbb{C}})$. The prequantization will be denoted as $(P_{\mathbb{C}}^{\ell}, \tilde{\theta}_{\mathbb{C}})$, and defined as follows.

Let $P_{\mathbb{C}}^{\ell} = \mathbb{C} \times \text{Spin}^c(2)$ be the the trivial principal $\text{Spin}^c(2)$ -principal bundle over \mathbb{C} with the non-trivial S^1 -action

$$S^1 \times P_{\mathbb{C}}^{\ell} \rightarrow P_{\mathbb{C}}^{\ell} \quad , \quad (e^{i\varphi}, (z, [a, w])) \mapsto (e^{-i\varphi} z, [x_{-\varphi/2} \cdot a, e^{-i\ell\varphi/2} \cdot w])$$

where $x_{\varphi} = \cos \varphi + \sin \varphi \cdot e_1 e_2 \in \text{Spin}(2)$. Note that since $\ell \in \mathbb{Z}$ is odd, this action is well defined. Next we define a connection

$$\theta_{\mathbb{C}}: TP_{\mathbb{C}}^{\ell} \rightarrow \mathfrak{spin}^c(2) = \mathfrak{spin}(2) \oplus \mathfrak{u}(1) .$$

Denote by $\pi_1: P_{\mathbb{C}}^{\ell} \rightarrow \mathbb{C}$ and $\pi_2: P_{\mathbb{C}}^{\ell} \rightarrow \text{Spin}^c(2)$ the projections, and by θ^R the right-invariant Maurer-Cartan form on $\text{Spin}^c(2)$. Then set

$$\theta_{\mathbb{C}}: TP_{\mathbb{C}}^{\ell} \rightarrow \text{Spin}^c(2) \quad , \quad \theta_{\mathbb{C}} = \pi_2^*(\theta^R) + \frac{1}{2} \pi_1^*(\bar{z} dz - z d\bar{z}) .$$

Note that $\pi_1^*(\bar{z} dz - z d\bar{z})$ takes values in $i\mathbb{R} = \mathfrak{u}(1) \subset \mathfrak{spin}^c(2)$, and that the connection $\theta_{\mathbb{C}}$ does not depend on ℓ .

Finally, let

$$\tilde{\theta}_{\mathbb{C}} = \frac{1}{2} \det_* \circ \theta_{\mathbb{C}} .$$

Claim 10.4.1. *For any odd $\ell \in \mathbb{Z}$, the pair $(P_{\mathbb{C}}^{\ell}, \tilde{\theta}_{\mathbb{C}})$ is an S^1 -equivariant spin^c -prequantization for $(\mathbb{C}, \omega_{\mathbb{C}})$.*

Proof. The 1-form $\theta_{\mathbb{C}}$ (and hence $\tilde{\theta}_{\mathbb{C}}$) is S^1 -invariant, since $\bar{z} dz - z d\bar{z}$ is an S^1 -invariant 1-form on \mathbb{C} , and since the group $\text{Spin}^c(2)$ is abelian. The 1-form $\tilde{\theta}_{\mathbb{C}}$ is given by

$$\tilde{\theta}_{\mathbb{C}} = \frac{1}{2} \det_* \circ \theta_{\mathbb{C}} = \frac{1}{2} \det_* \circ \pi_2^*(\theta^R) + \frac{1}{2} \pi_1^*(\bar{z} dz - z d\bar{z})$$

and therefore

$$d(\tilde{\theta}_{\mathbb{C}}) = 0 + \frac{1}{2} \pi_1^*(d\bar{z} \wedge dz - dz \wedge d\bar{z}) = \pi_1^*(-dz \wedge d\bar{z}) = \pi_1^*(-i \cdot \omega_{\mathbb{C}})$$

as needed. Finally, by Remark 10.3.2, we have $\tilde{\theta}_{\mathbb{C}}(\zeta_{P_{\mathbb{C}}^{\ell}}) = 0$ for all $\zeta \in \mathfrak{spin}(2)$.

□

Chapter 11

Cutting of a Spin^c Prequantization

The process of cutting consists of several steps: Taking the product, restricting and taking the quotient of spin^c structures. We start by discussing those constructions independently.

11.1 The product of two spin^c prequantizations

Let a Lie group G act by orientation preserving isometries on two oriented Riemannian manifolds M and N , of dimensions m and n , respectively. Given two equivariant spin^c structures P_M, P_N on M, N , we can take their ‘product’ as follows. First, note that $P_M \times P_N$ is a G -equivariant principal $\text{Spin}^c(m) \times \text{Spin}^c(n)$ -bundle on $M \times N$. Second, observe that $\text{Spin}^c(m)$ and $\text{Spin}^c(n)$ embed naturally as subgroups of $\text{Spin}^c(m+n)$, and thus give rise to a homomorphism

$$\text{Spin}^c(m) \times \text{Spin}^c(n) \rightarrow \text{Spin}^c(m+n) \quad , \quad (x, y) \mapsto x \cdot y .$$

This homomorphism is used to define a principal $\text{Spin}^c(m+n)$ -bundle on $M \times N$, denoted $P_{M \times N}$, as a fiber bundle associated to $P_M \times P_N$.

In the following claim, θ^L is the left invariant Maurer-Cartan 1-form on the group $\text{Spin}^c(m+n)$, and ω_M, ω_N are closed G -invariant two forms on M, N .

Claim 11.1.1. *Let (P_M, θ_M) and (P_N, θ_N) be two G -equivariant spin^c prequantizations for (M, ω_M) and (N, ω_N) , respectively.*

Let

$$P_{M \times N} = (P_M \times P_N) \times_{\text{Spin}^c(m) \times \text{Spin}^c(n)} \text{Spin}^c(m+n)$$

and

$$\theta_{M \times N} = \theta_M + \theta_N + \frac{1}{2} \det_* \circ \theta^L \in \Omega^1(P_{M \times N}; \mathfrak{u}(1)) .$$

Then $(P_{M \times N}, \theta_{M \times N})$ is a G -equivariant spin^c -prequantization for $(M \times N, \omega_M \oplus \omega_N)$, called the product of (P_M, θ_M) and (P_N, θ_N) .

Remark 11.1.1.

1. More specifically, the connection $\theta_{M \times N}$ is given by

$$\theta_{M \times N}(q_*(u, v, \xi^L)) = \theta_M(u) + \theta_N(v) + \frac{1}{2} \det_*(\xi)$$

where $u \in TP_M$, $v \in TP_N$, $\xi \in \mathfrak{spin}^c(m+n)$ and

$$q: P_M \times P_N \times \text{Spin}^c(m+n) \rightarrow P_{M \times N}$$

is the quotient map. This is well defined since θ_M and θ_N are spin^c -invariant.

2. The G -action on $M \times N$ can be taken to be either the diagonal action

$$g \cdot (x, y) = (g \cdot x, g \cdot y)$$

or the ‘action on M ’

$$g \cdot (x, y) = (g \cdot x, y)$$

and $(P_{M \times N}, \theta_{M \times N})$ will be a G -equivariant prequantization with respect to any of those actions.

3. The map $P_{M \times N} \rightarrow \text{SOF}(M \times N)$ is the natural one induced from $P_M \rightarrow \text{SOF}(M)$ and $P_N \rightarrow \text{SOF}(N)$, using the fact that

$$\text{SOF}(M \times N) \cong (\text{SOF}(M) \times \text{SOF}(N)) \times_{\text{SO}(m) \times \text{SO}(n)} \text{SO}(m+n) .$$

Proof. The connection $\theta_{M \times N}$ is G and $\text{Spin}^c(m+n)$ -invariant, since θ_M and θ_N have the same invariance properties. Moreover, since $d\theta^L = 0$, we get that

$$d(\theta_{M \times N}) = d(\theta_M) + d(\theta_N) = \pi^*(-i \cdot \omega_M \oplus \omega_N)$$

as needed, where $\pi: P_{M \times N} \rightarrow M \times N$ is the projection.

Finally, $\theta_{M \times N}(\zeta_{P_{M \times N}}) = 0$ for all $\zeta \in \mathfrak{spin}(m+n)$ since $\frac{1}{2} \det_*(\zeta) = 0$. \square

11.2 Restricting a spin^c prequantization

Assume that a Lie group G acts on an m dimensional oriented Riemannian manifold M by orientation preserving isometries. Let $Z \subset M$ be a G -invariant co-oriented submanifold of co-dimension 1. Then there is a natural map

$$i: \text{SOF}(Z) \rightarrow \text{SOF}(M) \quad , \quad i(f)(a_1, \dots, a_m) = f(a_1, \dots, a_{m-1}) + a_m \cdot v_p$$

where $f: \mathbb{R}^{m-1} \xrightarrow{\sim} T_p Z$ is a frame in $\text{SOF}(Z)$, and $v \in \Gamma(TM)$ is the vector field on Z of positive unit vectors orthogonal to TZ .

A G -equivariant spin^c -structure P on M can be restricted to Z , by setting

$$P_Z = i^*(P) \ ,$$

i.e., P_Z is the pullback under i of the circle bundle $P \rightarrow \text{SOF}(M)$. The relevant diagram is

$$\begin{array}{ccc} P_Z = i^*(P) & \xrightarrow{i'} & P \\ \downarrow & & \downarrow \\ \text{SOF}(Z) & \xrightarrow{i} & \text{SOF}(M) \\ \downarrow & & \downarrow \\ Z & \longrightarrow & M \end{array}$$

The principal action on $P_Z \rightarrow Z$ comes from the natural inclusion $\text{Spin}^c(m-1) \hookrightarrow \text{Spin}^c(m)$, and the G -action on P_Z is induced from the one on P .

Furthermore, if a connection 1-form θ is given on the circle bundle $P \rightarrow \text{SOF}(M)$, we can restrict it to a connection 1-form θ_Z on $P_Z \rightarrow \text{SOF}(Z)$ by letting

$$\theta_Z = (i')^*\theta .$$

Claim 11.2.1. *Let (P, θ) be a G -equivariant spin^c prequantization for (M, ω) (for a closed G -invariant two form ω), and $Z \subset M$ a co-oriented G -invariant submanifold of co-dimension 1. Then the pair (P_Z, θ_Z) is a G -equivariant spin^c -prequantization for $(Z, \omega|_Z)$.*

Proof.

$$d(\theta_Z) = (i')^*(d\theta) = (i')^*\pi^*(-i \cdot \omega) = \pi^*(-i \cdot \omega|_Z)$$

as needed, and

$$\theta_Z(\zeta_{P_Z}) = \theta(\zeta_P) = 0$$

for all $\zeta \in \mathfrak{spin}(m-1)$. □

11.3 Quotients of spin^c prequantization

Here is a general fact about connections on principal bundles and their quotients.

Claim 11.3.1. *Let H, K, G be three Lie groups, and $P \rightarrow X$ an H -equivariant and K -equivariant principal G -bundle. Assume that H acts freely on X , and that the H and K -actions on P commute (i.e., $h \cdot (k \cdot y) = k \cdot (h \cdot y)$ for all $h \in H, k \in K, y \in P$), then:*

1. $\pi: P/H \rightarrow X/H$ is a K -equivariant principal G -bundle.
2. If $\theta: TP \rightarrow \mathfrak{g}$ is a connection 1-form, and $q: P \rightarrow P/H$ is the quotient map, then $\theta = q^*(\bar{\theta})$ for some connection 1-form $\bar{\theta}: T(P/H) \rightarrow \mathfrak{g}$ if and only if θ is H -invariant, and $\theta(\xi_P) = 0$ for all $\xi \in \mathfrak{h}$.

Proof.

1. The surjection $P/H \rightarrow M/H$, induced from $\pi: P \rightarrow M$, and the right G -action on those quotient spaces are well defined since the left H -action commutes with the right G -action on P , and with the projection π .

To show that $P/H \rightarrow X/H$ is a principal G -bundle, it suffices to check that G acts freely on P/H . Indeed, if $[p] \in P/H$, $g \in G$ and $[p] \cdot g = [p]$, then this implies

$$[p \cdot g] = [p] \quad \Rightarrow \quad p \cdot g = h \cdot p$$

for some $h \in H$, which implies

$$\pi(p \cdot g) = \pi(h \cdot p) \quad \Rightarrow \quad \pi(p) = h \cdot \pi(p) .$$

But $H \curvearrowright X$ freely, and so $h = id$. Then $p \cdot g = p$, and since $P \curvearrowright G$ freely, we conclude that $g = id$, as needed.

It is easy to check that the K -action descends to $P/H \rightarrow X/H$, since it commutes with the H and the G -actions.

2. First assume that $\theta = q^*(\bar{\theta})$. If $h \in H$ acts on P , then

$$h^*\theta = h^*(q^*\bar{\theta}) = (q \circ h)^*\bar{\theta} = q^*\bar{\theta} = \theta$$

and so θ is H -invariant. Also, if $\xi \in \mathfrak{h}$, then clearly $q_*(\xi_P) = 0$, and hence $\theta(\xi_P) = (q^*\bar{\theta})(\xi_P) = 0$, as needed.

Conversely, assume that θ is H -invariant and that $\theta(\xi_P) = 0$ for all $\xi \in \mathfrak{h}$. For any $v \in TP$ define

$$\bar{\theta}(q_*v) = \theta(v) .$$

This is well defined: If $v \in T_yP$ and $v' \in T_{y'}P$ such that $q_*(v) = q_*(v')$, then $y' = h \cdot y$ for some $h \in H$, and we get that

$$\theta_{y'}(v') = \theta_{h \cdot y}(v') = h^*(\theta_y((h^{-1})_*v')) = \theta_y((h^{-1})_*v') .$$

Now observe that

$$q_*(v - (h^{-1})_*v') = q_*(v) - q_*(v') = 0 ,$$

and so $v - (h^{-1})_*v' = \xi_P|_x$ (for some $\xi \in \mathfrak{h}$) is in the vertical bundle of $P \rightarrow P/H$.

By assumption, $\theta(\xi_P) = 0$ and therefore $\theta_y(v) = \theta_{y'}(v')$, and $\bar{\theta}$ is well defined.

The map $\bar{\theta}: T(P/H) \rightarrow \mathfrak{g}$ is a 1-form. Smoothness is implied from the definition of the smooth structure on P/H . Also $\bar{\theta}$ is vertical and G -equivariant because θ is.

□

Now assume that Z is an n -dimensional oriented Riemannian manifold, and S^1 acts freely on Z by isometries. Let $P \rightarrow \text{SOF}(Z) \rightarrow Z$ be a G and S^1 -equivariant spin^c structure on Z . We would like to explain how one can get a G -equivariant spin^c -structure on Z/S^1 , induced from the given one on Z .

Denote by $\frac{\partial}{\partial\varphi} \in \text{Lie}(S^1) \simeq i\mathbb{R}$ the generator, and by $\left(\frac{\partial}{\partial\varphi}\right)_Z$ the corresponding vector field on Z . Define the normal bundle

$$V = \left[\left(\frac{\partial}{\partial\varphi} \right)_Z \right]^\perp \subset TZ$$

and an embedding $\eta: \text{SOF}(V) \rightarrow \text{SOF}(Z)$ as follows. If $f: \mathbb{R}^{n-1} \xrightarrow{\cong} V_x$ is a frame in $\text{SOF}(V)$, then $\eta(f): \mathbb{R}^n \xrightarrow{\cong} T_xZ$ will be given by $\eta(f)e_i = f(e_i)$ for $i = 1, \dots, n-1$, and $\eta(f)e_n$ is the unit vector in the direction of $\left(\frac{\partial}{\partial\varphi}\right)_{Z,x}$.

$$\begin{array}{ccc} \eta^*(P) & \xrightarrow{\eta'} & P \\ \downarrow & & \downarrow \\ \text{SOF}(V) & \xrightarrow{\eta} & \text{SOF}(Z) \\ \downarrow & & \downarrow \\ Z & \xlongequal{\quad} & Z \end{array}$$

To get a spin^c -structure on Z/S^1 , first consider the equivariant spin^c -structure on the vector bundle V

$$\eta^*(P) \rightarrow \text{SOF}(V) \rightarrow Z .$$

Once we take the quotient by the circle action, we get *the quotient* spin^c -structure on Z/S^1 , denoted by \bar{P} :

$$\bar{P} = \eta^*(P)/S^1 \rightarrow \text{SOF}(V)/S^1 \cong \text{SOF}(Z/S^1) \rightarrow Z/S^1 .$$

If an S^1 and $\text{Spin}^c(m)$ -invariant connection 1-form θ is given on the principal circle bundle $P \rightarrow \text{SOF}(Z)$, then $(\eta')^*\theta$ is a connection 1-form on the principal circle bundle $\eta^*(P) \rightarrow \text{SOF}(V)$.

The previous claim tells us exactly when the above connection 1-form will descend to a connection 1-form on the quotient bundle $\bar{P} \rightarrow \text{SOF}(Z/S^1)$. The following proposition summarizes the above construction and relates it to spin^c -prequantization.

Proposition 11.3.1. *Assume that the following data are given:*

1. *An n -dimensional Riemannian oriented manifold Z .*
2. *A real closed 2-form ω on Z .*
3. *Actions of a Lie group G and S^1 on Z , by orientation preserving and ω -invariant isometries.*
4. *A G and S^1 -equivariant spin^c prequantization (P, θ) on Z . Assume that the actions of G and S^1 on P and Z commute with each other.*

Also assume that the action $S^1 \curvearrowright Z$ is free.

Then, using the above notation, we have that:

1. *$\theta' = (\eta')^*\theta$ is a connection 1-form on the principal circle bundle $\pi: \eta^*(P) \rightarrow \text{SOF}(V)$, satisfying*

$$d\theta' = \pi^*(-i \cdot \omega) ,$$

and

$$\theta'(\zeta_{\eta^*(P)}) = 0 \quad \text{for all } \zeta \in \mathfrak{spin}(m-1).$$

2. If $\left(\frac{\partial}{\partial\varphi}\right)_{\eta^*(P)}$ is the vector field generated by the action $S^1 \curvearrowright \eta^*(P)$, and $q: \eta^*(P) \rightarrow \bar{P} = \eta^*(P)/S^1$ is the quotient map, then $\theta' = q^*(\bar{\theta})$ for some connection 1-form $\bar{\theta}$ on $\bar{P} \rightarrow \text{SOF}(Z/S^1)$ if and only if

$$\theta' \left[\left(\frac{\partial}{\partial\varphi} \right)_{\eta^*(P)} \right] = 0.$$

Moreover, in this case, $(\bar{P}, \bar{\theta})$ is a G -equivariant spin^c -prequantization for $G \curvearrowright (Z/S^1, \bar{\omega})$ (where $\omega = q^*(\bar{\omega})$).

Proof.

1. We have

$$d\theta' = (\eta')^* d\theta = (\eta')^* \circ \pi^*(-i \cdot \omega) = \pi^*(-i \cdot \omega)$$

and

$$\theta'(\zeta_{\eta^*(P)}) = \theta(\zeta_P) = 0$$

as needed.

2. The fact that $\theta' = q^*(\bar{\theta})$ if and only if

$$\theta' \left[\left(\frac{\partial}{\partial\varphi} \right)_{\eta^*(P)} \right] = 0$$

follows directly from Claim 11.3.1, since θ' is S^1 -invariant, and $\frac{\partial}{\partial\varphi}$ is a generator.

Finally, $(\bar{P}, \bar{\theta})$ is a prequantization, since

$$q^*(d\bar{\theta}) = d\theta' = \pi^*(-i \cdot \omega) = q^* \bar{\pi}^*(-i \cdot \bar{\omega}) \quad \Rightarrow \quad d\bar{\theta} = \bar{\pi}^*(-i \cdot \bar{\omega})$$

where $\bar{\pi}: \eta^*(P)/S^1 \rightarrow Z/S^1$ is the projection. Clearly, since all our objects are G -invariant, and all the actions commute, $(\bar{P}, \bar{\theta})$ is a G -equivariant prequantization.

□

Remark 11.3.1. When the condition in part (2) of the above proposition holds, we will say that *the prequantization (P, θ) for $G \curvearrowright (Z, \omega)$ descends to the prequantization $(\bar{P}, \bar{\theta})$ for $G \curvearrowright (Z/S^1, \bar{\omega})$.*

11.4 The cutting of a prequantization

In [4], Lerman describes a cutting construction for symplectic manifolds (M, ω) , endowed with a Hamiltonian circle action and a moment map $\Phi: M \rightarrow \mathfrak{u}(1)^*$, which goes as follows. If $\omega_{\mathbb{C}} = -i \cdot dz \wedge d\bar{z}$, then $(M \times \mathbb{C}, \omega \oplus \omega_{\mathbb{C}})$ is a symplectic manifold. The action

$$S^1 \times (M \times \mathbb{C}) \rightarrow M \times \mathbb{C} \quad , \quad (a, (m, z)) \mapsto (a \cdot m, a^{-1} \cdot z)$$

is Hamiltonian with moment map $\tilde{\Phi}(m, z) = \Phi(m) - |z|^2$.

If $\alpha \in \mathfrak{u}(1)^*$ and S^1 acts freely on $Z = \tilde{\Phi}^{-1}(\alpha)$, then α is a regular value of $\tilde{\Phi}$, and the (positive) cut space is defined by

$$M_{cut}^+ = \tilde{\Phi}^{-1}(\alpha)/S^1 = \{(m, z) \in M \times \mathbb{C} : \Phi(m) - |z|^2 = \alpha\} .$$

This is a symplectic manifold, with the symplectic form ω_{cut}^+ obtained by reduction, and S^1 acts on M_{cut}^+ by $a \cdot [m, z] = [a \cdot m, z]$. If M is also Riemannian oriented manifold, so is the cut space (but the natural inclusion $M_{cut}^+ \hookrightarrow M$ is not an isometry).

Assume that the following is given:

1. An m dimensional oriented Riemannian manifold.
2. A closed real two-form ω on M .
3. An action of S^1 on M by ω -invariant isometries.

4. An S^1 -equivariant spin^c prequantization $(P, \theta) = (P_M, \theta_M)$ for (M, ω) .

Recall that the action $S^1 \curvearrowright (M, \omega)$ is Hamiltonian, with moment map $\Phi: M \rightarrow \mathfrak{u}(1)^*$ determined by the equation

$$\pi^*(\Phi^\xi) = -i \cdot \iota_{\xi_P}(\theta) \quad , \quad \xi \in \mathfrak{u}(1)$$

where $\pi: P \rightarrow M$ is the projection, and ξ_P is the vector field on P generated by the S^1 -action (see Remark 10.3.1).

We want to cut the given spin^c prequantization. For that we choose $\alpha \in \mathfrak{u}(1)^*$ and set $Z = \Phi^{-1}(\alpha)$. We assume that S^1 acts on Z freely, and that α is a regular value of Φ (however, we do not assume that ω is nondegenerate). Our goal is to get a condition on α such that cutting along $Z = \Phi^{-1}(\alpha)$ is possible (i.e., such that a spin^c -prequantization on the cut space is obtained).

We proceed according to the following steps.

Step 1 Let S^1 act on the complex plane via

$$(a, z) \mapsto a^{-1} \cdot z \quad , \quad a \in S^1, z \in \mathbb{C} .$$

This action preserves the standard Riemannian structure and orientation, and the two form $\omega_{\mathbb{C}} = -i \cdot dz \wedge d\bar{z}$.

Fix an odd integer ℓ , and consider the S^1 -equivariant spin^c -prequantization $(P_{\mathbb{C}}^\ell, \tilde{\theta}_{\mathbb{C}})$ for $S^1 \curvearrowright (\mathbb{C}, \omega_{\mathbb{C}})$ defined in §10.4.

Step 2 Using Claim 11.1.1 we obtain an S^1 -equivariant spin^c prequantization

$$(P_{M \times \mathbb{C}}, \theta_{M \times \mathbb{C}}) \text{ for } S^1 \curvearrowright (M \times \mathbb{C}, \omega \oplus \omega_{\mathbb{C}}).$$

Step 3 Denote

$$\tilde{Z} = \{(m, z) : \Phi(m) - |z|^2 = \alpha\} \subset M \times \mathbb{C} .$$

This is an S^1 -invariant submanifold of codimension 1. By Claim 11.2.1, we get an S^1 -equivariant spin^c -prequantization $(P_{\tilde{Z}}, \theta_{\tilde{Z}})$ for $(\tilde{Z}, \omega_{\tilde{Z}})$, where $\omega_{\tilde{Z}}$ is the restriction of $\omega \oplus \omega_{\mathbb{C}}$ to \tilde{Z} .

Step 4 By Remark 11.1.1, the pair $(P_{\tilde{Z}}, \theta_{\tilde{Z}})$ is an S^1 -equivariant prequantization with respect to both the anti-diagonal and the action on M (in which S^1 acts on the M component via the given action, and on the \mathbb{C} component trivially).

Using the terminology introduced in Remark 11.3.1, we state our main theorem, which enable us to complete the process and get an equivariant prequantization on the (positive) cut space.

Theorem 11.4.1. *The S^1 -equivariant spin^c -prequantization $(P_{\tilde{Z}}, \theta_{\tilde{Z}})$ descends to an S^1 -equivariant spin^c -prequantization on $(\tilde{Z}/S^1 = M_{cut}^+, \omega_{cut}^+)$ if and only if*

$$\alpha = \frac{\ell}{2} \in \mathfrak{u}(1)^* = \mathbb{R}$$

Proof. By Proposition 11.3.1, $(P_{\tilde{Z}}, \theta_{\tilde{Z}})$ will descend to a prequantization on the cut space, if and only if

$$\theta_{\tilde{Z}} \left[\left(\frac{\partial}{\partial \varphi} \right)_{\eta^*(P_{\tilde{Z}})} \right] = 0 .$$

This is the same as requiring that $\theta_{\tilde{Z}}$, when restricted to $\eta^*(P_{\tilde{Z}})$, vanishes:

$$\theta_{\tilde{Z}} \left[\left(\frac{\partial}{\partial \varphi} \right)_{P_{\tilde{Z}}} \right] \Big|_{\eta^*(P)} = 0 ,$$

which is equivalent to

$$\theta_{M \times \mathbb{C}} \left[\left(\frac{\partial}{\partial \varphi} \right)_{P_{M \times \mathbb{C}}} \right] = 0 \quad \text{on} \quad \eta^*(P_{\tilde{Z}}) .$$

Now using the formula for $\theta_{M \times \mathbb{C}}$, we get that

$$\theta_M \left(\left(\frac{\partial}{\partial \varphi} \right)_{P_M} \right) + \theta_{\mathbb{C}} \left(\left(\frac{\partial}{\partial \varphi} \right)_{P_{\mathbb{C}}} \right) = 0$$

It is not hard to show that at a point $(z, [A, w]) \in P_{\mathbb{C}}^{\ell} = \mathbb{C} \times \text{Spin}^c(2)$, we have

$$\left(\frac{\partial}{\partial \varphi} \right)_{P_{\mathbb{C}}^{\ell}} = i \cdot \left[\bar{z} \frac{\partial}{\partial \bar{z}} - z \frac{\partial}{\partial z} \right] + \nu|_{[A, w]}$$

where $\nu|_{[A, w]}$ is the vector field on $\text{Spin}^c(2)$ generated by the element

$$\nu = -\frac{1}{2} e_1 e_2 - \frac{i \cdot \ell}{2} \in \mathfrak{spin}^c(2) .$$

Therefore one computes that

$$\theta_{\mathbb{C}} \left(\left(\frac{\partial}{\partial \varphi} \right)_{P_{\mathbb{C}}^{\ell}} \right) = -i \cdot \left(|z|^2 + \frac{\ell}{2} \right)$$

On the other hand, by the condition defining our moment map, we have that

$$\theta_M \left(\left(\frac{\partial}{\partial \varphi} \right)_{P_M} \right) = i \cdot \pi^* (\Phi^{\partial/\partial \varphi})$$

where $\pi: P \rightarrow M$ is the projection.

Combining the above we see that $(P_{\tilde{Z}}, \theta_{\tilde{Z}})$ descends to an S^1 -equivariant spin^c -prequantization on $(\tilde{Z}/S^1 = M_{cut}^+, \omega_{cut}^+)$ if and only if (on $\eta^*(P_{\tilde{Z}})$):

$$\pi^* (\Phi^{\partial/\partial \varphi}) - |z|^2 - \frac{\ell}{2} = 0 .$$

But on the manifold \tilde{Z} we have $\Phi(m) - |z|^2 = \alpha$. and hence the last equality is equivalent to

$$\alpha - \frac{\ell}{2} = 0 ,$$

as needed. □

Remark 11.4.1. We can also construct a spin^c -prequantization for the negative cut space $(M_{cut}^-, \omega_{cut}^-)$ as follows. Recall that M_{cut}^- is defined as the quotient

$$\{(m, z) \in M \times \mathbb{C} : \Phi(m) + |z|^2 = \alpha\} / S^1 ,$$

where the S^1 -action on $M \times \mathbb{C}$ is taken to be the diagonal action, and ω_{cut}^- is defined as before by reduction. The two form on \mathbb{C} is taken to be $i dz \wedge d\bar{z}$, and the spin^c -prequantization for \mathbb{C} is defined using the connection

$$\theta_{\mathbb{C}} = \pi^*(\theta^R) - \frac{1}{2}(\bar{z}dz - zd\bar{z}) .$$

The S^1 -action on $P_{\mathbb{C}}^{\ell}$ will be given by

$$S^1 \times P_{\mathbb{C}}^{\ell} \rightarrow P_{\mathbb{C}}^{\ell} \quad , \quad (e^{i\varphi}, (z, [a, w])) \mapsto (e^{i\varphi}z, [x_{\varphi/2} \cdot a, e^{-i\ell\varphi/2} \cdot w])$$

(see §10.4).

Other than that, the construction is carried out as for the positive cut space, and we can prove a theorem that will assert that $\alpha = \ell/2$, if the cutting is to be done along the level set $\Phi^{-1}(\alpha)$ of the moment map.

Chapter 12

An Example - The Two Sphere

In this chapter we discuss in detail spin^c prequantizations and cutting for the two-sphere.

12.1 Prequantizations for the two-sphere

The two-sphere will be thought of as a submanifold of \mathbb{R}^3 :

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$$

with the outward orientation and natural Riemannian structure induced from the inner product in \mathbb{R}^3 . Fix a real number c , and let $\omega = c \cdot A$, where A is the area form on the two-sphere

$$A = j^*(x \, dy \wedge dz + y \, dz \wedge dx + z \, dx \wedge dy) ,$$

and where $j: S^2 \hookrightarrow \mathbb{R}^3$ is the inclusion. Note that ω is a symplectic form if and only if $c \neq 0$.

For any real φ define

$$C_\varphi = \begin{pmatrix} \cos \varphi & -\sin \varphi & 0 \\ \sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} ,$$

and let S^1 act on S^2 via rotations around the z -axis, i.e.,

$$(e^{i\varphi}, v) \mapsto C_\varphi \cdot v \quad , \quad v \in S^2 .$$

In Chapter 8, we constructed all S^1 -equivariant spin^c -structures over the S^1 -manifold S^2 (up to equivalence). Let us review the main ingredients here.

First, the *trivial spin^c -structure* P_0 is given by the following diagram.

$$\begin{array}{ccccc} S^1 \times \text{Spin}^c(3) & \longrightarrow & P_0 = \text{Spin}^c(3) & \longleftarrow & \text{Spin}^c(3) \times \text{Spin}^c(2) \\ \downarrow & & \Lambda \downarrow & & \downarrow \\ S^1 \times SO(3) & \longrightarrow & SO(3) & \longleftarrow & SO(3) \times SO(2) \\ \downarrow & & \pi \downarrow & & \\ S^1 \times S^2 & \longrightarrow & S^2 & & \end{array}$$

In this diagram we use the fact that the frame bundle of S^2 is isomorphic to $SO(3)$. The projection π is given by

$$A \mapsto A \cdot x$$

where $x = (0, 0, 1)$ is the north pole, and the map Λ is the obvious one.

The horizontal maps describe the S^1 and the principal actions: S^1 and $SO(2)$ act on $SO(3)$ by left and right multiplication by C_φ , respectively. The principal action of $\text{Spin}^c(2)$ on $\text{Spin}^c(3)$ is just right multiplication, and the S^1 action on $\text{Spin}^c(3)$ is given by

$$(e^{i\varphi}, [A, z]) \mapsto [x_{\varphi/2} \cdot A, e^{i\varphi} \cdot z]$$

where $x_{\varphi/2} = \cos \varphi + \sin \varphi \cdot e_1 e_2 \in \text{Spin}(3)$. We can turn this spin^c structure into a spin^c -prequantization as follows. Let $\omega_0 = 0$ the zero two form on S^2 , and consider the 1-form

$$\theta_0 = \frac{1}{2} \det_* \circ \theta^R : T\text{Spin}^c(3) \rightarrow \mathfrak{u}(1) = i\mathbb{R}$$

where θ^R is the right-invariant Maurer-Cartan form on $\text{Spin}^c(3)$ and the map \det was defined in §10.3. Clearly, (P_0, θ_0) is an S^1 -equivariant spin^c -prequantization for (S^2, ω_0) .

Next, we construct all S^1 -equivariant line bundles over S^2 .

Claim 12.1.1. *Given a pair of integers (k, n) , define an S^1 -equivariant complex Hermitian line bundle $L_{k,n}$ as follows:*

1. *As a complex line bundle,*

$$L_{k,n} = S^3 \times_{S^1} \mathbb{C} ,$$

where S^1 acts on \mathbb{C} with weight n and on $S^3 \subset \mathbb{C}^2$ by

$$S^1 \times S^3 \rightarrow S^3 \quad , \quad (a, (z, w)) \mapsto (az, aw) .$$

2. *The circle group S^1 acts on $L_{k,n}$ by*

$$S^1 \times L_{k,n} \rightarrow L_{k,n} \quad , \quad (e^{i\varphi}, [(z, w), u]) \mapsto [(e^{i\varphi/2}z, e^{-i\varphi/2}w), e^{i(n+2k)\varphi/2} \cdot u] .$$

Then every equivariant line bundle over S^2 is equivariantly isomorphic to $L_{k,n}$ for some integers k, n .

For the proof, see Claim 8.2.1 (where slightly different notation is used).

To get all spin^c structures on S^2 , we need to twist P_0 with the $U(1)$ -bundle $U(L_{k,n})$ associated to $L_{k,n}$ for some $k, n \in \mathbb{Z}$. Thus define

$$P_{k,n} = P_0 \times_{U(1)} U(L_{k,n}) .$$

The principal $\text{Spin}^c(2)$ -action is given coming from the action on P_0 , and the left S^1 -action is induced from the diagonal action.

We now define a connection

$$\theta_n : TP_{k,n} \rightarrow i\mathbb{R}$$

on the $U(1)$ bundle $P_{k,n} \rightarrow SO(3) = \text{SOF}(S^2)$, which will not depend on k , as follows:

$$\theta_n = \theta_0 + \frac{n}{2} (-\bar{z} dz + z d\bar{z} - \bar{w} dw + w d\bar{w}) + u^{-1} du$$

where $(z, w) \in S^3 \subset \mathbb{C}^2$ are coordinates on S^3 and $u^{-1} du$ is the Maurer-Cartan form on the S^1 component of $U(L_{k,n}) = S^3 \times_{S^1} S^1$.

One can compute

$$d\theta_n = n(dz \wedge d\bar{z} + dw \wedge d\bar{w}) = \pi^*(-in/2 \cdot A)$$

and hence if we define $\omega_n = \frac{n}{2} \cdot A$ then $(P_{k,n}, \theta_n)$ is a spin^c prequantization for (S^2, ω_n) .

Let P_{det} be the $U(1)$ -bundle associated to the determinant line bundle of a spin^c structure. We proved in Chapter 8, that the determinant line bundle of any spin^c structure on the two-sphere is isomorphic to $L_{2k+1,2n}$ (see Remark 8.2.2), and hence has a square root (as a non-equivariant line bundle). Using this fact and the construction of $(P_{k,n}, \theta_n)$ above, we prove:

Claim 12.1.2. *The S^1 -manifold $(S^2, \omega = c \cdot A)$ is spin^c -prequantizable (i.e., admits an S^1 -equivariant spin^c -prequantization) if and only if $2c \in \mathbb{Z}$.*

Proof. Assume that (P, θ) is a spin^c -prequantization for (S^2, ω) . Then, by Claim 10.3.2, $\theta = \frac{1}{2}q^*(\bar{\theta})$ for some connection 1-form $\bar{\theta}$ on the principal $U(1)$ -bundle $p: P_{det} \rightarrow S^2$, where $q: P \rightarrow P/Spin(2) = P_{det}$ is the quotient map. Since (P, θ) is a spin^c prequantization, we have

$$d\theta = \pi^*(-i \cdot \omega) \quad \Rightarrow \quad q^*\left(\frac{1}{2}d\bar{\theta}\right) = q^*p^*(-i \cdot \omega) \quad \Rightarrow \quad \frac{1}{2}d\bar{\theta} = p^*(-i \cdot \omega)$$

which implies

$$d\bar{\theta} = p^*(-2i \cdot \omega) .$$

This means that $[-2i \cdot \omega]$ is the curvature class of the determinant line bundle of P . According to the above remark, P_{det} is a square, and hence the class

$$\frac{1}{2}[-2i \cdot \omega] = [-i \cdot \omega]$$

is a curvature class of a line bundle over S^2 . This forces $[\omega]$ to be integral (Weyl's theorem - page 172 in [1]), i.e.,

$$\int_{S^2} \omega \in 2\pi\mathbb{Z} \quad \Rightarrow \quad 2c \in \mathbb{Z}$$

and the conclusion follows.

Conversely, assume that $2c \in \mathbb{Z}$. Then, as mentioned above, $(P_{k,2c}, \theta_{2c})$ (for any $k \in \mathbb{Z}$) is a spin^c prequantization for $(S^2, c \cdot A)$ as needed. \square

Let us now compute the moment map

$$\Phi: S^2 \rightarrow \mathfrak{u}(1)^* = \mathbb{R}$$

for $(S^2, n/2 \cdot A)$ (for $n \in \mathbb{Z}$) determined by the prequantization $(P_{k,n}, \theta_n)$. Recall that

$$\theta_n = \theta_0 + \frac{n}{2} (-\bar{z} dz + z d\bar{z} - \bar{w} dw + w d\bar{w}) + u^{-1} du .$$

It is straightforward to show that the vector field, generated by the left S^1 -action on $P_{k,n}$ is

$$\left(\frac{\partial}{\partial \varphi} \right)_{P_{k,n}} = \frac{i}{2} \frac{\partial}{\partial v} - \frac{i}{2} \left(-\bar{z} \frac{\partial}{\partial \bar{z}} + z \frac{\partial}{\partial z} + \bar{w} \frac{\partial}{\partial \bar{w}} - w \frac{\partial}{\partial w} \right) + \frac{i}{2} (n + 2k) \frac{\partial}{\partial u}$$

where $\frac{\partial}{\partial v}$ is the vector field on P_0 generated by the S^1 -action.

Now compute

$$\begin{aligned} \theta_n \left(\left(\frac{\partial}{\partial \varphi} \right)_{P_{k,n}} \right) &= \frac{i}{2} - \frac{in}{4} (-\bar{z}z + z(-\bar{z}) - \bar{w}w) + \frac{i}{2} (n + 2k) = \\ &= \frac{i}{2} [n(|z|^2 - |w|^2) + n + 2k + 1] \end{aligned}$$

and thus Φ is given by

$$\Phi([z, w]) = -i \cdot \theta_n \left(\left(\frac{\partial}{\partial \varphi} \right)_{P_{k,n}} \right) = \frac{n}{2} (|z|^2 - |w|^2 + 1) + k + \frac{1}{2}$$

Remark 12.1.1. Observe that for $[z, w] \in S^2 = \mathbb{C}P^1$, the quantity $|z|^2 - |w|^2$ represents the third coordinate x_3 (i.e., the height) on the unit sphere (this is part of the Hopf-fibration).

Since $-1 \leq x_3 \leq 1$, we have (for $n \geq 0$):

$$k + \frac{1}{2} \leq \Phi \leq n + k + \frac{1}{2}$$

and hence the image of the moment map is the closed interval

$$\left[k + \frac{1}{2}, n + k + \frac{1}{2} \right]$$

if $n \geq 0$ or

$$\left[n + k + \frac{1}{2}, k + \frac{1}{2} \right]$$

if $n \leq 0$.

12.2 Cutting a prequantization on the two-sphere

Fix an S^1 -equivariant spin^c -prequantization $(P_{k,n}, \theta_n)$ for (S^2, ω_n) , where $\omega_n = \frac{n}{2} \cdot A$ (A is the area form on the two-sphere) and $n \neq 0$.

The corresponding moment map, as computed above, is

$$\Phi: S^2 \rightarrow \mathbb{R} \quad , \quad \Phi([z, w]) = \frac{n}{2} (|z|^2 - |w|^2 + 1) + k + \frac{1}{2}$$

We would like to cut this prequantization along a level set $\Phi^{-1}(\alpha)$ of the moment map. By Theorem 11.4.1 we must have

$$\alpha = \frac{\ell}{2}$$

for some odd integer ℓ , and the cutting has to be done using the spin^c -structure $(P_{\mathbb{C}}^{\ell}, \theta_{\mathbb{C}})$ on $(\mathbb{C}, \omega_{\mathbb{C}})$ (see §10.4).

In Chapter 8 we performed the cutting construction for the two-sphere in the case where $\ell = 1$. In this case we showed that the spin^c structures obtained for the cut spaces are

$$(P_{k,n})_{cut}^+ = P_{0,k+n} \quad , \quad (P_{k,n})_{cut}^- = P_{k,-k} .$$

These computations can be modified for an arbitrary ℓ to get

$$(P_{k,n})_{cut}^+ = P_{(\ell-1)/2, k+n-(\ell-1)/2} \quad , \quad (P_{k,n})_{cut}^- = P_{k, -k+(\ell-1)/2} .$$

Recall that the cut spaces obtained in this case are symplectomorphic to two-spheres (if $\ell/2$ is strictly between $k + \frac{1}{2}$ and $n + k + \frac{1}{2}$). Using this identification we have:

Claim 12.2.1. *If the symplectic manifold (S^2, ω_n) , endowed with the Hamiltonian S^1 -action*

$$(e^{i\varphi}, v) \mapsto C_\varphi \cdot v$$

and the above moment map Φ is being cut along the level set $\Phi^{-1}(\ell/2)$, then the reduced two-forms on the cut spaces are

$$\omega_{cut}^+ = \omega_{k+n+(1-\ell)/2} \quad \text{and} \quad \omega_{cut}^- = \omega_{-k+(\ell-1)/2} .$$

Here we assume that $\ell/2$ is strictly between $k + \frac{1}{2}$ and $n + k + \frac{1}{2}$.

Proof. Let us concentrate on the positive cut space. We will use cylindrical coordinates (ϕ, h) to describe the point

$$(x, y, z) = (\sqrt{1-h^2} \cos \phi, \sqrt{1-h^2} \sin \phi, h)$$

on the unit sphere S^2 . The positive cut space is obtained by reduction. The relevant diagram is

$$\begin{array}{ccc} \tilde{Z} & \xrightarrow{i} & S^2 \times \mathbb{C} \\ p \downarrow & & \\ \tilde{Z}/S^1 \cong S^2 & & \end{array}$$

Recall that

$$\tilde{Z} = \{((\phi, h), u) \in S^2 \times \mathbb{C} : \Phi(\phi, h) - |u|^2 = \ell/2\}$$

and that the two-form on $S^2 \times \mathbb{C}$ is

$$\omega_n + \omega_{\mathbb{C}} = \frac{n}{2} \cdot A - i du \wedge d\bar{u} .$$

The map p is given by

$$((\phi, h), u = r e^{-i\alpha}) \mapsto \left(\phi + \alpha, \frac{2n}{2n + 2k + 1 - \ell} (h - 1) + 1 \right) .$$

The pullback of the area form on S^2 via p is

$$A' = (d\phi + d\alpha) \wedge \frac{2n}{2n + 2k + 1 - \ell} dh = \frac{2n}{2n + 2k + 1 - \ell} (d\phi \wedge dh - \frac{2i}{n} du \wedge d\bar{u}) ,$$

and thus the pullback of $\omega_{k+n+(1-\ell)/2}$ via p is

$$\frac{k+n+(1-\ell)/2}{2} \cdot A' = \frac{n}{2}A - i du \wedge d\bar{u} = \omega_n + \omega_{\mathbb{C}}$$

as needed.

A similar proof is obtained for the negative cut space. \square

To complete the cutting, we need to find out what are the corresponding connections $\theta^{\pm} = (\theta_n)_{cut}^{\pm}$ on $(P_{k,n})_{cut}^{\pm}$. Instead of going through the cutting process of a connection, we proceed as follows (for the positive cut space).

We know that $((P_{k,n})_{cut}^+, \theta^+)$ must be a spin^c -prequantization for

$$((S^2)_{cut}^+, \omega_{cut}^+) = (S^2, \omega_{k+n+(1-\ell)/2}) .$$

This means that

$$d\theta^+ = d\theta_{k+n+(1-\ell)/2}$$

which implies that

$$\theta^+ - \theta_{k+n+(1-\ell)/2} = \pi^* \beta$$

for some closed one-form $\beta \in \Omega^1(S^2; \mathfrak{u}(1))$. But then $\beta = df$ is also exact since S^2 is simply connected. We conclude that

$$\theta^+ = \theta_{k+n+(1-\ell)/2} + d(\pi^*(f)) ,$$

thus, the bundle $((P_{k,n})_{cut}^+, \theta^+)$ is gauge equivalent to $((P_{k,n})_{cut}^+, \theta_{k+n+(1-\ell)/2})$.

A similar argument can be carried out for the negative cut space. We summarize:
The cutting of (S^2, ω_n) along the level set $\Phi^{-1}(\ell/2)$ yields two spin^c prequantizations:

$$(P_{k,-k+(\ell-1)/2}, \theta_{-k+(\ell-1)/2}) \quad \text{for} \quad ((S^2)_{cut}^- = S^2, \omega_{-k+(\ell-1)/2})$$

and

$$(P_{(\ell-1)/2, k+n+(1-\ell)/2}, \theta_{k+n+(1-\ell)/2}) \quad \text{for} \quad ((S^2)_{cut}^+ = S^2, \omega_{k+n+(1-\ell)/2}) .$$

Chapter 13

Prequantizing $\mathbb{C}P^n$

In this chapter we construct a spin^c -prequantization for the complex projective space $\mathbb{C}P^n$ (with the standard Riemannian structure coming from the Kähler structure). For $n = 1$ we have shown that a two form ω on $\mathbb{C}P^1 \cong S^2$ is spin^c prequantizable if and only if $\frac{1}{2\pi}\omega$ is integral (i.e., $\int_{\mathbb{C}P^1} \frac{1}{2\pi}\omega \in \mathbb{Z}$ - see Claim 12.1.2). This is not true in general. We will prove that for an even n , if $(\mathbb{C}P^n, \omega)$ is spin^c prequantizable then $\frac{1}{2\pi}\omega$ will not be integral. This is an important difference between spin^c -prequantization and the geometric prequantization scheme of Kostant and Souriau (an excellent reference for geometric quantization is [13]).

From now on, fix a positive integer n . Points in $\mathbb{C}P^n$ will be written as $[v]$, where $v \in S^{2n+1} \subset \mathbb{C}^{n+1}$. The Fubini-Study form ω_{FS} on $\mathbb{C}P^n$ will be normalized (as in [14, page 261]) so that $\int_{\mathbb{C}P^1} \omega_{FS} = 1$ (where $\mathbb{C}P^1$ is naturally embedded into $\mathbb{C}P^n$). We describe our construction in steps. For simplicity, we discuss the non-equivariant case (where the acting group G is the trivial group), but our results will apply to the equivariant case as well. Also, $|\cdot|$ will denote the determinant of a matrix.

Step 1 - Constructing a Spin^c structure.

The group $SU(n+1)$ acts transitively on $\mathbb{C}P^n$ via

$$SU(n+1) \times \mathbb{C}P^n \rightarrow \mathbb{C}P^n \quad , \quad (A, [v]) \mapsto [A \cdot v] .$$

Let $p = e_{n+1} \in \mathbb{C}^{n+1}$ denote the unit vector $(0, \dots, 0, 1)$. The stabilizer of p under the $SU(n+1)$ -action is

$$H = S(U(n) \times U(1)) = \left\{ \begin{pmatrix} B & 0 \\ 0 & |B|^{-1} \end{pmatrix} : B \in U(n) \right\} \subset SU(n+1)$$

and so $\mathbb{C}P^n \cong SU(n+1)/H$ via

$$[A] \mapsto [A \cdot p] .$$

The tangent space $T_{[p]}\mathbb{C}P^n$ can be identified with \mathbb{C}^n and then the isotropy representation is given by

$$\sigma: H \rightarrow U(n) \quad , \quad \sigma \left(\begin{pmatrix} B & 0 \\ 0 & |B|^{-1} \end{pmatrix} \right) = |B| \cdot B .$$

The frame bundle of $\mathbb{C}P^n$ can then be described as an associated bundle (using $U(n) \subset SO(2n)$):

$$SOF(\mathbb{C}P^n) = SU(n+1) \times_{\sigma} SO(2n) .$$

The map

$$f: U(n) \rightarrow SO(2n) \times S^1 \quad , \quad A \mapsto (A, |A|)$$

has a lift $F: U(n) \rightarrow \text{Spin}^c(2n)$ (see [1, page 27] for an explicit formula for F). Using that, we define

$$P = SU(n+1) \times_{\tilde{\sigma}} \text{Spin}^c(2n)$$

where $\tilde{\sigma} = F \circ \sigma: H \rightarrow \text{Spin}^c(2n)$.

Thus we get a spin^c -structure $P \rightarrow SOF(\mathbb{C}P^n) \rightarrow \mathbb{C}P^n$ on the n -dimensional complex projective space.

Step 2 - Constructing a connection on $P \rightarrow SOF(\mathbb{C}P^n)$.

Let $\theta^R: TSU(n+1) \rightarrow \mathfrak{su}(n+1)$ be the right-invariant Maurer-Cartan form, and define

$$\chi: \mathfrak{su}(n+1) \rightarrow \mathfrak{h} = Lie(H) \quad , \quad \begin{pmatrix} A & * \\ * & -tr(A) \end{pmatrix} \mapsto \begin{pmatrix} A & 0 \\ 0 & -tr(A) \end{pmatrix} .$$

Since χ is an equivariant map under the adjoint action of H , we conclude that

$$\chi \circ \theta^R: TSU(n+1) \rightarrow \mathfrak{h}$$

is a connection 1-form on the (right-) principal H -bundle

$$SU(n+1) \rightarrow \mathbb{C}P^n = SU(n+1)/H .$$

This induces a connection 1-form on the principal $Spin^c(2n)$ -bundle $P \rightarrow \mathbb{C}P^n$:

$$\hat{\theta}: TP \rightarrow \mathfrak{spin}^c(2n) .$$

After composing $\hat{\theta}$ with the projection

$$\frac{1}{2}det_*: \mathfrak{spin}^c(2n) = \mathfrak{spin}(2n) \oplus \mathfrak{u}(1) \rightarrow \mathfrak{u}(1) = i\mathbb{R}$$

We get a connection 1-form $\theta = \frac{1}{2}det_* \circ \hat{\theta}$ on the principal $U(1)$ -bundle $P \rightarrow SOF(\mathbb{C}P^n)$.

In fact, here is an explicit formula for the connection θ :

If $\xi = \begin{pmatrix} A & * \\ * & -tr(A) \end{pmatrix} \in \mathfrak{su}(n+1)$, $\zeta \in \mathfrak{spin}^c(2n)$, ξ^R and ζ^L are the corresponding

vector fields on $SU(n+1)$ and $Spin^c(2n)$, and

$$q: SU(n+1) \times Spin^c(2n) \rightarrow P$$

is the quotient map, then a direct computation gives

$$\theta(q_*(\xi^R + \zeta^L)) = \frac{n+1}{2} \cdot tr(A) + \frac{1}{2}det_*(\zeta) .$$

Note that if $\zeta \in \mathfrak{spin}(2n)$, then $\theta(q_*(\zeta^L)) = 0$.

Step 3 - Computing the curvature of θ .

Using the formula

$$d\theta(V, W) = V\theta(W) - W\theta(V) - \theta([V, W])$$

for any two vector fields V, W on P , we can compute the curvature $d\theta$ of the connection θ . We obtain the following:

If $\xi_1, \xi_2 \in \mathfrak{su}(n+1)$, $\zeta_1, \zeta_2 \in \mathfrak{spin}^c(2n)$, and

$$[\xi_1, \xi_2] = \begin{pmatrix} X & * \\ * & * \end{pmatrix} \in \mathfrak{su}(n+1)$$

then we have

$$d\theta(q_*(\xi_1^R + \zeta_1^L), q_*(\xi_2^R + \zeta_2^L)) = -\frac{n+1}{2} \cdot \text{tr}(X).$$

Let ω be the real two form on $\mathbb{C}P^n$ for which

$$d\theta = \pi^*(-i \cdot \omega).$$

In fact

$$\omega = -\frac{n+1}{2} \cdot 2\pi \omega_{FS}$$

where ω_{FS} is the Fubini-Study form. To see this, it is enough, by $SU(n+1)$ -invariance of ω and ω_{FS} , to show the above equality at one point (for instance, at $[p] \in \mathbb{C}P^n$).

Recall that the cohomology class of ω_{FS} generates the integral cohomology of $\mathbb{C}P^n$, i.e., $\int_{\mathbb{C}P^1} \omega_{FS} = 1$. This immediately implies that our two form ω is integral if and only if n is odd, and we have:

(P, θ) is a spin^c -prequantization for $(\mathbb{C}P^n, \omega)$.

Remark 13.0.1. It is not hard to conclude, that a spin^c prequantizable two form ω on $\mathbb{C}P^n$ is integral if and only if n is odd. In fact, Proposition D.43 in [2], together with Claim 10.3.2 imply the following:

For an odd n , a two-form ω on $\mathbb{C}P^n$ is spin^c prequantizable if and only if $\frac{1}{2\pi}\omega$ is integral, i.e., $[\frac{1}{2\pi}\omega] \in \mathbb{Z}[\omega_{FS}]$.

For an even n , a two-form ω on $\mathbb{C}P^n$ is spin^c prequantizable if and only if $[\frac{1}{2\pi}\omega] \in (\mathbb{Z} + \frac{1}{2})[\omega_{FS}]$.

Part III

A Universal Property of the Groups

Spin^c and Mp^c

Chapter 14

Introduction to Part III

Around 1928, while studying the motion of a free particle in special relativity, P.A.M Dirac raised the problem of finding a square root to the three dimensional Laplacian, acting on smooth functions on \mathbb{R}^3 . Finding a square root was necessary in order to study such a system in the quantum mechanics setting. His assumptions were that such a square root ought to be a first order differential operator with constant coefficients.

It is well known that in the Euclidean space \mathbb{R}^n , the problem of finding such a square root involves Clifford algebras and their representations. This square root is often called *a Dirac operator*, and it acts on the representation space for the Clifford algebra (also called *the space of spinors*).

The transition from the flat Euclidean space to a general Riemannian manifold is not obvious. There is no representation of the group $SO(n)$ on the space of spinors which is compatible with the Clifford algebra action. This means that in order to generalize the construction of the Dirac operator to Riemannian manifolds, we must introduce additional structure. More precisely, we need to lift the Riemannian structure (where the group involved is $SO(n)$) to a ‘better’ group G .

It is known that the construction of the Dirac operator can be carried out if our manifold has a spin, a spin^c, or an almost complex structure.

In this part we answer the question: what is the best (or universal) structure that enables the above process? We show that the group $Spin^c(n)$ (or a non-compact variant of it) is a universal solution to our problem, in the sense that any other solution will factor uniquely through the $spin^c$ one. This suggests that $spin^c$ structures are the natural ones to consider while quantizing the classical energy observable.

It is important to note that $spin^c$ structures are equivalent to having an irreducible Clifford module on the manifold. This fact appears as Theorem 2.11 in [17]. This is another hint for the universality of the group $spin^c$.

Interestingly, a similar problem can be stated in the symplectic case, and the universal solution involves Mp^c structures, discussed in [16]. The universality statement and the proof are almost identical to those in the Riemannian case.

This part is organized as follows. First we introduce the problem of finding a square root for the n -dimensional Laplacian acting on smooth (complex-valued) functions on \mathbb{R}^n . This will motivate the definition of Clifford algebras and the study of their representations. Then we generalize the problem to an arbitrary oriented Riemannian manifold, and explain why more structure is needed in order to define the Dirac operator. Next, we state our main theorem about the universality of the (non-compact variant of the) $spin^c$ group, and deduce a few corollaries. In the last section, we prove a similar result in the symplectic setting. Namely, the universality of the metaplectic^c group.

This work was motivated by the introduction of [1], where the problem of defining a Dirac operator for an arbitrary oriented Riemannian manifold is discussed, and the necessity of additional structure is mentioned. For the study of symplectic Dirac operators, we refer to [15], which is the ‘symplectic analogue’ of [1]. Both [1] and [15] are excellent resources.

Chapter 15

The Euclidean Case and Clifford Algebras

Consider the negative Laplacian acting on smooth complex valued functions on the n -dimensional Euclidean space

$$\Delta: C^\infty(\mathbb{R}^n; \mathbb{C}) \rightarrow C^\infty(\mathbb{R}^n; \mathbb{C}) \quad , \quad \Delta = - \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2} \quad ,$$

and suppose we are interested in finding a square root for Δ , i.e., we seek an operator

$$P: C^\infty(\mathbb{R}^n; \mathbb{C}) \rightarrow C^\infty(\mathbb{R}^n; \mathbb{C})$$

with $P^2 = \Delta$. Motivated by physics, we assume that P is a first order differential operator with constant coefficients, i.e., that

$$P = \sum_{j=1}^n \gamma_j \frac{\partial}{\partial x_j} \quad , \quad \gamma_j \in \mathbb{C} .$$

A simple computation shows that such a P cannot exist unless $n = 1$, and then $P = \pm i \frac{\partial}{\partial x}$. Indeed, the condition $P^2 = \Delta$ implies that

$$P^2 = \sum_{j,l=1}^n \gamma_j \gamma_l \frac{\partial^2}{\partial x_j \partial x_l} = - \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$$

and hence

$$\gamma_j \gamma_l + \gamma_l \gamma_j = 0 \quad , \quad \gamma_j^2 = -1 \quad \text{for all } j \neq l$$

which is impossible, unless $n = 1$ and $\gamma_1 = \pm i$.

One way to modify this problem is to observe that the commutativity of complex numbers ($\gamma_j \gamma_l = \gamma_l \gamma_j$) is the property that made this construction impossible. Therefore, we hope to be able to find such a P if the γ_j 's are taken to be matrices, instead of complex numbers.

Thus, fix an integer $k > 1$, define the *vector valued Laplacian* as

$$\underline{\Delta}: C^\infty(\mathbb{R}^n; \mathbb{C}^k) \rightarrow C^\infty(\mathbb{R}^n; \mathbb{C}^k) \quad , \quad \underline{\Delta}(f_1, \dots, f_k) = (\Delta f_1, \dots, \Delta f_k) \ ,$$

and look for a square root

$$\underline{P}: C^\infty(\mathbb{R}^n; \mathbb{C}^k) \rightarrow C^\infty(\mathbb{R}^n; \mathbb{C}^k) \ .$$

If we assume, as before, that

$$\underline{P} = \sum_{j=1}^n \gamma_j \frac{\partial}{\partial x_j} \quad \text{with} \quad \gamma_j \in M_{k \times k}(\mathbb{C}) \ ,$$

then $\underline{P}^2 = \underline{\Delta}$ if and only if

$$\gamma_j \gamma_l + \gamma_l \gamma_j = 0 \quad , \quad \gamma_j^2 = -1 \quad \text{for} \quad j \neq l \ .$$

Those relations are precisely the ones used to define *the Clifford algebra* associated to the vector space \mathbb{R}^n . Here is a more common and general definition of this concept.

Definition 15.0.1. For a finite dimensional vector space V over $\mathbb{K} = \mathbb{R}$ or \mathbb{C} , and a symmetric bilinear map $B: V \times V \rightarrow \mathbb{K}$, define *the Clifford algebra*

$$Cl(V, B) = T(V)/I(V, B)$$

where $T(V)$ is the tensor algebra of V , and $I(V, B)$ is the ideal generated by

$$\{v \otimes v - B(v, v) \cdot 1 : v \in V\} \ .$$

Remark 15.0.2.

1. If e_1, \dots, e_n is an orthogonal basis for V , then $Cl(V, B)$ is the algebra generated by $\{e_j\}$ with relations

$$e_j e_l + e_l e_j = 0 \quad , \quad e_j^2 = B(e_j, e_j) \quad \text{for} \quad j \neq l .$$

2. If \langle , \rangle is the standard inner product on \mathbb{R}^n , then denote

$$C_n = Cl(\mathbb{R}^n, - \langle , \rangle) \quad \text{and} \quad C_n^c = C_n \otimes \mathbb{C} .$$

Example 15.0.1. For $n = 3$, let

$$\gamma_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \gamma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \gamma_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} .$$

Then $\{\gamma_1, \gamma_2, \gamma_3\}$ satisfy the required relations, and $\underline{P} = \gamma_1 \frac{\partial}{\partial x_1} + \gamma_2 \frac{\partial}{\partial x_2} + \gamma_3 \frac{\partial}{\partial x_3}$ will be a square root of $\underline{\Delta} = -\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2}$.

The above discussion suggests that we look for a representation

$$\rho: C_n \rightarrow \text{End}(\mathbb{C}^k) \cong M_{k \times k}(\mathbb{C})$$

of the Clifford algebra C_n . It will be even better if we can find an irreducible one (since every representation of C_n is a direct sum of irreducible ones - see Proposition I.5.4 in [3]). Once we fix such a representation, we can set

$$\gamma_j = \rho(e_j) \quad , \quad \underline{P} = \sum_{j=1}^n \gamma_j \frac{\partial}{\partial x_j}$$

where $\{e_j\}$ is the standard basis for \mathbb{C}^k . The operator \underline{P} , called a *Dirac operator*, will then be a square root of $\underline{\Delta}$.

Here is a known fact about representations of complex Clifford algebras (proofs can be found in [1] and in [3]).

Proposition 15.0.1. *Any irreducible complex representation of C_n has dimension $2^{\lfloor n/2 \rfloor}$ (where $\lfloor n/2 \rfloor$ is the floor of $n/2$). Up to equivalence, there are two irreducible representations if n is odd, and only one if n is even.*

We conclude that finding a square root for $\underline{\Delta}$ is always possible. It is defined using an irreducible representation of C_n . Note that a choice is to be made if n is odd.

Chapter 16

Manifolds

16.1 The problem

We would like to generalize our previous construction from the Euclidean space to a smooth n dimensional oriented Riemannian manifold (M, g) . More generally, we look for a complex Hermitian vector bundle $S \rightarrow M$ and a smooth bundle map

$$\rho: Cl(TM, -g) \oplus S \rightarrow S \quad , \quad (\alpha, v) \mapsto \rho_x(\alpha)v$$

for $\alpha \in Cl(T_x M)$, $v \in S_x$, which restricts to an irreducible representation

$$\rho_x: Cl(T_x M, -g_x) \rightarrow End(S_x)$$

on the fibers of S . The notation $Cl(TM, -g)$ stands for *the Clifford bundle* of (M, g) , and \oplus above denotes the Whitney sum. That is, the vector bundle whose fibers are the Clifford algebra of the tangent space, with respect to the symmetric bilinear map $-g$.

Once such a pair (S, ρ) is found, we can choose a Hermitian connection ∇ on S , and define a Dirac operator acting on smooth sections of S , as follows. Choose a local orthonormal frame $\{e_j\}$ and let

$$\underline{P}: \Gamma(S) \rightarrow \Gamma(S) \quad , \quad \underline{P}(s) = \sum_{j=1}^n \rho(e_j) [\nabla_{e_j} s] \quad .$$

It turns out that \underline{P} is independent of the local frame, and thus gives rise to a globally defined operator on sections of S .

16.2 The search for the vector bundle S

If no additional structure is introduced on our manifold M , then we may try to construct the vector bundle S as an associated bundle to the bundle $SOF(M)$ of oriented orthonormal frames on M . This means that we try to take

$$S = SOF(M) \times_{SO(n)} \mathbb{C}^k$$

where $k = 2^{\lfloor n/2 \rfloor}$ and $SO(n)$ acts on \mathbb{C}^k via a representation

$$\epsilon: SO(n) \rightarrow \text{End}(\mathbb{C}^k) .$$

We can use an irreducible representation

$$\rho: C_n \rightarrow \text{End}(\mathbb{C}^k)$$

in order to define an action of the Clifford bundle $Cl(TM)$ on S as follows.

For any $x \in M$, $\alpha \in T_x M \subset Cl(T_x M, -g_x)$, $v \in \mathbb{C}^k$ and a frame $f: \mathbb{R}^n \xrightarrow{\simeq} T_x M$ in $SOF_x(M)$, let α act on $[f, v] \in S_x$ by

$$(\alpha, [f, v]) \mapsto [f, \rho(f^{-1}(\alpha))v] .$$

This will be a well defined action on S if and only if

$$[f \circ A, \rho((f \circ A)^{-1}\alpha)v] = [f, \rho(f^{-1}(\alpha))(\epsilon(A)v)]$$

for any $A \in SO(n)$. This is equivalent to

$$\epsilon(A) \circ \rho(A^{-1}y) = \rho(y) \circ \epsilon(A)$$

where $y = f^{-1}(\alpha)$, and is an equality between linear endomorphisms of \mathbb{C}^k . To summarize, we look for a representation $\epsilon: SO(n) \rightarrow \text{End}(\mathbb{C}^k)$ with the property that

$$\rho(Ay) = \epsilon(A) \circ \rho(y) \circ \epsilon(A)^{-1} \quad (16.1)$$

for all $A \in SO(n)$ and $y \in \mathbb{R}^n$.

Unfortunately:

Claim 16.2.1. *For $n \geq 3$, there is no representation $\epsilon: SO(n) \rightarrow \text{End}(\mathbb{C}^k)$ satisfying Equation (16.1) for all A and y .*

The proof will follow from a more general statement later (see Claim 18.0.1).

16.3 Introducing additional structure

It seems that in order to construct a vector bundle S over an n dimensional oriented Riemannian manifold (M, g) , on which the Clifford bundle $Cl(TM)$ acts irreducibly, we will have to introduce new structure on our manifold: we will need to lift the structure group from $SO(n)$ to a ‘better’ group G . Here is what we mean by *lifting the structure group*.

Definition 16.3.1. For an n dimensional oriented Riemannian manifold (M, g) , a *lifting of the structure group to a Lie group G* is a principal G -bundle $\pi: P \rightarrow M$, together with a group homomorphism $p: G \rightarrow SO(n)$ and a smooth map $\pi_1: P \rightarrow SOF(M)$ such that

$$\pi_1(x \cdot g) = \pi_1(x) \cdot p(g) \quad \text{for} \quad x \in P, g \in G,$$

and such that $\pi = \pi_2 \circ \pi_1$ (where $\pi_2: SOF(M) \rightarrow M$ is the projection).

In other words, we require that the following diagram will commute, and $\pi = \pi_2 \circ \pi_1$.

$$\begin{array}{ccc}
 P & \longleftarrow & P \times G \\
 \pi_1 \downarrow & & \downarrow \pi_1 \times p \\
 SOF(M) & \longleftarrow & SOF(M) \times SO(n) \\
 \pi_2 \downarrow & & \\
 M & &
 \end{array}$$

Once we have such a lift, we can try to construct our vector bundle of rank $k = 2^{\lfloor n/2 \rfloor}$ as

$$S = P \times_G \mathbb{C}^k ,$$

where G acts on \mathbb{C}^k via a representation $\epsilon: G \rightarrow End(\mathbb{C}^k)$.

This will work if the action of $Cl(TM)$ on S , given by

$$(\alpha, [\tilde{f}, v]) \mapsto [\tilde{f}, \rho(f^{-1}(\alpha))v] ,$$

is well defined. Here $\alpha \in Cl(T_x M)$, $\tilde{f} \in P_x$, $v \in \mathbb{C}^k$, and $f = p(\tilde{f}): \mathbb{R}^n \rightarrow T_x M$ is a frame in $SOF_x(M)$. As before, ρ is a fixed irreducible complex representation of C_n .

Equation (16.1), which state the condition ϵ has to satisfy, becomes

$$\rho(p(A)y) = \epsilon(A) \circ \rho(y) \circ \epsilon(A)^{-1} \tag{16.2}$$

for all $y \in \mathbb{R}^n$ and $A \in G$.

To summarize, we look for a Lie group G , and a representation $\epsilon: G \rightarrow End(\mathbb{C}^k)$ for which Equation (16.2) is satisfied for all A and y .

Chapter 17

The Universality Theorem

Given an irreducible representation $\rho: C_n \rightarrow \text{End}(\mathbb{C}^k)$ ($k = 2^{\lfloor n/2 \rfloor}$), we look for a Lie group G , a representation $\epsilon: G \rightarrow \text{End}(\mathbb{C}^k)$, and a homomorphism $p: G \rightarrow SO(n)$ for which Equation (16.2) is satisfied. Note that this problem is of algebraic flavour and does not involve the manifold, the tangent bundle or the Clifford bundle. Thus, our problem is reduced to one where the unknowns are a Lie group, a representation and a group homomorphism.

As we will see, there is more than one solution to this problem, but only one (up to a certain equivalence) which is universal in the sense that every other solution will factor through the universal one. In this universal solution, the group is

$$G = (\text{Spin}(n) \times \mathbb{C}^\times) / K$$

where $\text{Spin}(n)$ is the double cover of $SO(n)$ and K is the two element subgroup generated by $(-1, -1)$. This is a noncompact group.

Another way to define this group is as the set of all elements of the form

$$c \cdot v_1 v_2 \cdots v_l \in C_n^c = C_n \otimes \mathbb{C}$$

where $c \in \mathbb{C}^\times$, $l \geq 0$ is even, and each $v_j \in \mathbb{R}^n$ is of (Euclidean) norm 1.

For each element $x \in G$ and $y \in \mathbb{R}^n \subset \mathbb{C}_n^c$, we have $Ad_x(y) = x \cdot y \cdot x^{-1} \in \mathbb{R}^n$, and the map

$$y \in \mathbb{R}^n \mapsto Ad_x(y) \in \mathbb{R}^n$$

is in $SO(n)$. This defines a group homomorphism

$$\lambda^c: G \rightarrow SO(n) \quad , \quad x \mapsto Ad_x$$

(see [1] for details).

Finally, note that any $B \in SO(n)$ acts on the Clifford algebra C_n^c in a natural way. This action is induced from the standard action of $SO(n)$ on \mathbb{R}^n . Furthermore, Equation (16.2) is satisfied for all $y \in \mathbb{R}^n$ and $A \in G$ if and only if it is satisfied for all $y \in C_n^c$ and $A \in G$.

Now we can state the universality property of the group G .

Theorem 17.0.1. *Fix an irreducible complex representation ρ of C_n^c , and let $k = 2^{\lfloor n/2 \rfloor}$.*

Then:

1. *For $G = (Spin(n) \times \mathbb{C}^\times) / K$, $p = Ad: G \rightarrow SO(n)$ and $\epsilon = \rho|_G: G \rightarrow End(\mathbb{C}^k)$, we have*

$$\rho(p(A)y) = \epsilon(A) \circ \rho(y) \circ \epsilon(A)^{-1}$$

for all $y \in C_n^c$ and $A \in G$.

2. *If G' is a Lie group, $p': G' \rightarrow SO(n)$ a group homomorphism, and $\epsilon': G' \rightarrow End(\mathbb{C}^k)$ a representation, such that*

$$\rho(p'(A)y) = \epsilon'(A) \circ \rho(y) \circ \epsilon'(A)^{-1}$$

for all $y \in C_n^c$ and $A \in G'$, then there is a unique homomorphism $f: G' \rightarrow G$ such that

$$p' = p \circ f \quad \text{and} \quad \epsilon' = \epsilon \circ f .$$

Remark 17.0.1.

1. The group G acts on C_n^c via

$$(A, y) \mapsto p(A)y$$

and on $End(\mathbb{C}^k)$ via

$$(A, \varphi) \mapsto \epsilon(A) \circ \varphi \circ \epsilon(A)^{-1} .$$

Therefore, in part (1) of the theorem we claim that ρ is G -equivariant.

2. Part (2) of the theorem implies that the following two diagrams are commutative.

$$\begin{array}{ccc} & & G \\ & \nearrow f & \downarrow p \\ G' & \xrightarrow{p'} & SO(n) \end{array} \qquad \begin{array}{ccc} & & G \\ & \nearrow f & \downarrow \epsilon \\ G' & \xrightarrow{\epsilon'} & End(\mathbb{C}^k) \end{array}$$

Proof.

1. For any $A \in G$ and $y \in C_n^c$ we have

$$\begin{aligned} \rho(p(A)y) &= \rho(Ad_A(y)) = \rho(A \cdot y \cdot A^{-1}) = \rho(A) \cdot \rho(y) \cdot \rho(A^{-1}) = \\ &= \epsilon(A) \circ \rho(y) \circ \epsilon(A)^{-1} \end{aligned}$$

2. Fix an element $g \in G'$, and choose an element $A \in Spin(n)$ for which $p(A) = Ad_A = p'(g)$. We claim that the endomorphism

$$\rho(A^{-1}) \circ \epsilon'(g): \mathbb{C}^k \rightarrow \mathbb{C}^k$$

is a nonzero (complex) multiple of the identity. To see this, start from the given equality

$$\rho(p'(g)y) = \epsilon'(g) \circ \rho(y) \circ \epsilon'(g)^{-1}$$

which is equivalent to

$$\epsilon'(g) \circ \rho(y) = \rho(Ad_A(y)) \circ \epsilon'(g)$$

and to

$$[\rho(A^{-1}) \circ \epsilon'(g)] \circ \rho(y) = \rho(y) \circ [\rho(A^{-1}) \circ \epsilon'(g)]$$

for all $y \in C_n^c$.

It is known that any irreducible complex representation of C_n^c must be onto, and hence the last equality means that $\rho(A^{-1}) \circ \epsilon'(g)$ commutes with all endomorphisms of \mathbb{C}^k , and thus must be a multiple of the identity (it is nonzero since it is invertible).

Write $\rho(A^{-1}) \circ \epsilon'(g) = c \cdot I$ for $c \in \mathbb{C}^\times$ and define

$$f: G' \rightarrow G \quad , \quad g \mapsto [A, c] \in G .$$

This map is well defined. It is a group homomorphism since if $g_j \in G'$ (for $j = 1, 2$), $A_j \in Spin(n)$ with $p'(g_j) = Ad_{A_j}$ and $c_j \in \mathbb{C}^\times$ satisfy

$$c_j \cdot I = \rho(A_j^{-1}) \circ \epsilon'(g_j)$$

then we have

$$p(g_1 g_2) = Ad_{A_1 A_2}$$

and

$$c_1 c_2 \cdot I = \rho((A_1 A_2)^{-1}) \circ \epsilon'(g_1 g_2) .$$

This implies that $f(g_1 g_2) = f(g_1) f(g_2)$.

Also we have

$$p'(g) = Ad_A = Ad_{c \cdot A} = p(f(g))$$

and

$$\epsilon'(g) = c \cdot \rho(A) = \epsilon(c \cdot A) = \epsilon(f(g))$$

for all $g \in G'$ as needed.

It is not hard to see that our construction implies also the uniqueness of the map f . After all, if such an f exists, and for $g \in G'$ we write $f(g) = [A, c] \in G$, then

$p(A) = Ad_A = p'(g)$, which means that A is determined up to sign. Furthermore, the relation $\epsilon'(g) = \epsilon(f(g))$ implies that

$$\rho(A^{-1}) \circ \epsilon'(g) = \epsilon([1, c]) = c \cdot I ,$$

which determines the value of c . Therefore $f(g) = [A, c]$ is uniquely determined by our conditions.

□

Remark 17.0.2.

1. The triple $(G = (Spin(n) \times \mathbb{C}^\times) / K, p, \epsilon)$ is the only universal solution up to equivalence. More precisely, if (G', p', ϵ') is another universal solution, then there is a unique isomorphism $\varphi: G' \rightarrow G$ satisfying $\epsilon' = \epsilon \circ \varphi$ and $p' = p \circ \varphi$.
2. There is a natural Hermitian product on the representation space \mathbb{C}^k with respect to which ρ is unitary. If we require that ϵ' will be unitary, then universal solution will involve the (compact) group

$$Spin^c(n) = (Spin(n) \times U(1)) / K .$$

The universality statement and the proof are almost identical to the noncompact case.

3. It is important to note that although the Dirac operator is a square root of the Laplacian in the case of \mathbb{R}^n , this is no longer true in the manifold case. The Dirac operator, whose definition was outlined in §16.1, will be related to the Laplacian via the Schrödinger-Lichnerowicz formula, which involves the scalar curvature of the manifold, and the curvature of the Hermitian connection on the vector bundle S (see §3.3 in [1]).

Chapter 18

Some Corollaries

Denote again by $\rho: C_n \rightarrow \text{End}(\mathbb{C}^k)$ ($k = 2^{\lfloor n/2 \rfloor}$) an irreducible representation, $G = (\text{Spin}(n) \times \mathbb{C}^\times) / K$, and by $p: G \rightarrow \text{SO}(n)$ the natural homomorphism. Then Theorem 17.0.1 implies the following.

Corollary 18.0.1. *A Lie group G' and a homomorphism $p': G' \rightarrow \text{SO}(n)$ give rise to a bijection between*

$$\mathcal{A} = \left\{ \begin{array}{l} \text{Representations } \epsilon': G' \rightarrow \text{End}(\mathbb{C}^k) \text{ satisfying} \\ \epsilon'(g) \circ \rho(y) = \rho(p'(g)y) \circ \epsilon'(g) \text{ for all } g \in G', y \in \mathbb{R}^n \end{array} \right\}$$

and

$$\mathcal{B} = \{ \text{Homomorphisms } f: G' \rightarrow G \text{ such that } p' = p \circ f \}$$

Proof. Part (2) in Theorem 17.0.1 provides a function $f \in \mathcal{B}$ for every $\epsilon' \in \mathcal{A}$. Conversely, if $f \in \mathcal{B}$, then $\epsilon' = \epsilon \circ f$ is in \mathcal{A} , by part (1) of Theorem 17.0.1. \square

The above corollary provides an easy criterion for checking whether a lifting of the structure to a group G' will enable us to construct an irreducible Clifford bundle action

on a vector bundle associated with this lifting. We give a few examples in the following claim.

Claim 18.0.1. *If $G' = U(n/2)$ (for an even n) or $G' = Spin(n)$, then there exist a homomorphism $p': G' \rightarrow SO(n)$ and a representation $\epsilon': G' \rightarrow End(\mathbb{C}^k)$ for which $\epsilon'(g) \circ \rho(y) = \rho(p'(g)y) \circ \epsilon'(g)$ for all $g \in G'$, $y \in \mathbb{R}^n$.*

If $G' = SO(n)$ ($n \geq 3$) and $p': G' \rightarrow SO(n)$ is the identity, then there is no ϵ' satisfying the latter equality.

Proof. For $G' = Spin(n)$ take p' to be the double cover of $SO(n)$, $f: Spin(n) \rightarrow G$ the inclusion, and use Corollary 18.0.1.

For $G' = U(n/2)$, take p' to be the standard inclusion $U(n/2) \subset SO(n)$. It is possible to define $f: U(n/2) \rightarrow G$ such that $p' = p \circ f$ (see page 27 in [1]). By the above corollary, the conclusion follows.

Finally, for $G' = SO(n)$ and $p' = Id$, if such an ϵ' would exist, the corollary implies that there is an $f: SO(n) \rightarrow G$ for which $p \circ f = Id$. This is impossible since the fundamental group of $SO(n)$ is \mathbb{Z}_2 and of G is \mathbb{Z} . □

The above claim implies some well known facts: Every spin and every almost complex manifold is also a $spin^c$ manifold in a natural way. Also, an irreducible Clifford module cannot be defined as a tensor bundle (i.e., as a vector bundle associated with the frame bundle of the manifold).

Chapter 19

The Symplectic Case

For the symplectic group, a similar problem can be stated. The universal group in this case will be the complexified metaplectic group $Mp^c(n) = Mp(n) \times_{\mathbb{Z}_2} U(1)$, if we demand unitary representations, or $Mp(n) \times_{\mathbb{Z}_2} \mathbb{C}^\times$ otherwise. In this section we outline the setting in this case, and prove a similar universality statement.

19.1 Symplectic Clifford algebras

Let V be a real vector space of dimension $2n$. If $B: V \times V \rightarrow \mathbb{R}$ is a symmetric bilinear form on V , then the ideal (in the tensor algebra $T(V)$) generated by expressions of the form

$$v \cdot v - B(v, v) \cdot 1 \quad , \quad v \in V \quad (19.1)$$

is the same one generated by

$$v \cdot u + u \cdot v - 2 \cdot B(u, v) \quad , \quad u, v \in V . \quad (19.2)$$

Suppose now that $\omega: V \times V \rightarrow \mathbb{R}$ is a symplectic (i.e., an antisymmetric and non-degenerate bilinear) form on V . Since $\omega(v, v) = 0$ for all $v \in V$, we would better modify (19.2) and define the symplectic Clifford algebra as follows. We follow [15] and omit the coefficient ‘2’ in our definition.

Definition 19.1.1. *The symplectic Clifford algebra associated with the symplectic vector space (V, ω) is defined as*

$$Cl^s(V, \omega) = T(V)/I(V, \omega)$$

where $I(V, \omega)$ is the ideal generated by

$$\{v \cdot w - w \cdot v + \omega(v, w) \cdot 1 : v, w \in V\} .$$

Remark 19.1.1. If $V = \mathbb{R}^{2n}$ and ω is the standard symplectic form, given by

$$\omega(x, y) = \sum_{j=1}^n x_j y_{n+j} - x_{n+j} y_j \quad , \quad x, y \in \mathbb{R}^{2n} ,$$

then we denote

$$Cl_n^s = Cl^s(\mathbb{R}^{2n}, \omega) \quad \text{and} \quad \mathbb{C}l_n^s = Cl_n^s \otimes \mathbb{C} .$$

The symplectic Clifford algebra in this case is also called the *Weyl algebra*, and is useful since its generators satisfy relations which are similar to the relations satisfied by the position and momentum operators in quantum mechanics (see §1.4 in [15]).

Denote by $\mathcal{S}(\mathbb{R}^n)$ the Schwartz space of rapidly decreasing complex-valued smooth functions on \mathbb{R}^n . If e_1, \dots, e_{2n} is the standard basis for \mathbb{R}^{2n} , then define a linear action

$$\rho: \mathbb{R}^{2n} \rightarrow \text{End}(\mathcal{S}(\mathbb{R}^n))$$

by assigning

$$\begin{aligned} \rho(e_j)f &= i \cdot x_j f & \text{for } j &= 1, \dots, n \\ \rho(e_j)f &= \frac{\partial f}{\partial x_j} & \text{for } j &= n+1, \dots, 2n \end{aligned}$$

and extend by linearity. This action extends (see §1.4 in [15]) to a linear map

$$Cl_n^s \rightarrow \text{End}(\mathcal{S}(\mathbb{R}^n))$$

which is *not* an algebra homomorphism.

For each $v \in \mathbb{R}^{2n}$, $\rho(v)$ can be regarded as a continuous operator on the Schwartz space. We call the map ρ *Clifford multiplication*.

19.2 The metaplectic representation

The metaplectic group $Mp(n)$ will play the role that the spin group $Spin(n)$ played in the Riemannian case. The symplectic group is

$$Sp(n) = \{A \in GL(2n, \mathbb{R}) : \omega(Av, Aw) = \omega(v, w)\}$$

where ω is the standard symplectic form on \mathbb{R}^{2n} . This is a connected and non-compact Lie group.

The fundamental group of $Sp(n)$ is isomorphic to \mathbb{Z} , and thus $Sp(n)$ has a unique connected double cover, which is denoted by $Mp(n)$. Denote by

$$p: Mp(n) \rightarrow Sp(n)$$

the covering map, and by $-1 \in Mp(n)$ the nontrivial element in $Ker(p)$.

Define

$$G = (Mp(n) \times \mathbb{C}^\times) / K$$

where $K = \{(1, 1), (-1, -1)\}$. The covering map extends to a map (also denoted by p)

$$G \rightarrow Sp(n) \quad , \quad [A, z] \mapsto p(A) .$$

There is an important infinite dimensional unitary representation of the metaplectic group on the Hilbert space $L^2(\mathbb{R}^n)$, which is called *the metaplectic representation*. We denote it by

$$\mathfrak{m}: Mp(n) \rightarrow U(L^2(\mathbb{R}^n))$$

where $U(L^2(\mathbb{R}^n))$ is the group of unitary operators on $L^2(\mathbb{R}^n)$. For the construction of \mathfrak{m} , see [15] and references therein. This representation has many interesting properties, but all we need here is the facts that the Schwartz space $\mathcal{S}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$ is an invariant subspace for \mathfrak{m} , and is dense in $L^2(\mathbb{R}^n)$.

We extend \mathfrak{m} to a representation of the group $G = (Mp(n) \times \mathbb{C}^\times) / K$, by

$$\epsilon: G \rightarrow End(L^2(\mathbb{R}^n)) \quad , \quad \epsilon([A, z]) = z \cdot \mathfrak{m}(A) .$$

19.3 The universality of the metaplectic group

Now we can state the universality theorem (for the group G), which turns out to be almost identical to the corresponding theorem in the Riemannian case.

Theorem 19.3.1. *Let $\rho: \mathbb{R}^{2n} \rightarrow \text{End}(\mathcal{S}(\mathbb{R}^n))$ be the Clifford multiplication map. Then:*

1. *For $G = (Mp(n) \times \mathbb{C}^\times) / K$, $p: G \rightarrow Sp(n)$ and $\epsilon: G \rightarrow \text{End}(L^2(\mathbb{R}^n))$ defined above, we have*

$$\rho(p(A)y) = \epsilon(A) \circ \rho(y) \circ \epsilon(A)^{-1}$$

for all $y \in \mathbb{R}^{2n}$ and $A \in G$ (i.e., ρ is G -equivariant).

This is an equality of operators on the Schwartz space $\mathcal{S}(\mathbb{R}^n)$.

2. *If G' is a Lie group, $p': G' \rightarrow Sp(n)$ a group homomorphism, and $\epsilon': G' \rightarrow \text{End}(\mathcal{S}(\mathbb{R}^n))$ a continuous representation, such that*

$$\rho(p'(A)y) = \epsilon'(A) \circ \rho(y) \circ \epsilon'(A)^{-1}$$

for all $y \in \mathbb{R}^{2n}$ and $A \in G'$, then there is a unique homomorphism $f: G' \rightarrow G$ such that

$$p' = p \circ f \quad \text{and} \quad \epsilon' = \epsilon \circ f .$$

Proof.

1. For $Mp(n)$, this is proved in Lemma 1.4.4 in [15]. The proof for G follows.
2. To prove the second part, we follow the same idea as in the Riemannian case. Fix an element $g \in G'$, and choose an element $A \in Mp(n)$ for which $p(A) = p'(g)$. We show that the endomorphism

$$D = \epsilon(A^{-1}) \circ \epsilon'(g): \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^n)$$

is a nonzero complex multiple of the identity. Once this is done, the rest of the proof will be identical to the proof of Theorem 17.0.1, part (2).

By assumption, we have

$$\epsilon(A) \circ \rho(y) \circ \epsilon(A)^{-1} = \epsilon'(g) \circ \rho(y) \circ \epsilon'(g)^{-1}$$

which is equivalent to

$$\rho(y) \circ D = D \circ \rho(y)$$

for all $y \in \mathbb{R}^{2n}$. From the definition of ρ we conclude that D is a continuous operator on the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ which commutes with all multiplication and derivative operators:

$$f \mapsto x_j \cdot f \quad \text{and} \quad f \mapsto \frac{\partial f}{\partial x_j} .$$

Such an operator must be a complex multiple of the identity. This follows from the fact that the map ρ gives rise to an irreducible representation of the symplectic Clifford algebra Cl_n^s on the space $L^2(\mathbb{R}^n; \mathbb{C})$. For a proof of this fact for $n = 1$ see Theorem 3 (page 44) in [14]. The n -dimensional case follows.

□

Remark 19.3.1. As in the Riemannian case, if we require that ϵ' will be a unitary representation, then the group G in Theorem 19.3.1 will be replaced with $(Mp(n) \times U(1)) / K$.

Remark 19.3.2. The construction of a Dirac operator in the Riemannian case was motivated by the search for a square root for the (negative) Laplacian. One may wonder what is the symplectic analog of the Dirac and the Laplacian operators. In Chapter 5 of [15] symplectic Dirac and associated second order operators are discussed. However, it is not clear to me if the *search for a square root* in the Riemannian case has a (satisfactory) symplectic analogue.

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